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Geometric Analysis and Interaction of Weyl and P_1 Curvature Tensors in Generalized Recurrent Cartan Finsler Spaces

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Abstract— The geometric relationship between the P_1 -curvature tensor and Weyl's projective curvature tensor in Cartan Finsler spaces is examined in this study. We generate new identities and establish recurrence conditions characterizing their interaction in generalized recurrent Finsler spaces using higher-order Cartan covariant derivatives. Additionally, we examine related tensorial structures, such as the scalar curvature, Ricci tensor, deviation tensor, and torsion tensor, which yield a number of characterization findings. These discoveries add to a better understanding of curvature tensor interactions in non-Riemannian geometric contexts and extend previous findings in Finsler geometry.

Keywords— Cartan Finsler spaces, Weyl projective curvature tensor, P_1 -curvature tensor, generalized recurrent spaces, Cartan connection, curvature tensors.

List Of Symbols

- 1) W.C.T.: Weyl Curvature Tensor.
- 2) P_1 -C.T.: P_1 -Curvature Tensor.
- 3) G.R.S.: Generalized Recurrent Spaces.

I. Introduction

Riemannian geometry naturally extends to Finsler geometry, where the metric is dependent on both direction and position. This framework has garnered significant interest in contemporary differential geometry and has shown itself to be very useful in characterizing anisotropic geometric formations ([31], [32] and [36]).

In order to comprehend the intrinsic geometry of Finsler manifolds, curvature tensors are essential. Weyl's projective curvature tensor is particularly significant among them because of its invariance characteristics and function in describing projective structures. Concurrently, the P_1 -curvature tensor offers further information about the geometric behavior of Finsler spaces and the tensorial structures that go along with them ([1], [2] and [3]).

The research of generalized recurring structures in Finsler geometry has advanced significantly in recent years, revealing intricate relationships between scalar curvature, curvature tensors, torsion tensors, and Ricci tensors. Nevertheless, nothing is known about the geometric interaction between the P_1 -curvature tensor and Weyl's projective curvature tensor in the context of Cartan Finsler spaces ([4], [5] and [6]).

Inspired by this gap, this study examines how Weyl and P_1 curvature tensors interact in generalized recurrent Cartan Finsler spaces. We generate new identities and define recurrence conditions governing their relationship using higher-order Cartan covariant derivatives. Additionally, we

derive a number of characterization results involving related tensorial quantities such as scalar curvature, torsion, deviation, and the Ricci tensor ([7], [8], [9] and [14]).

These results expand on current theories of recurrence in Finsler spaces and deepen our understanding of curvature structures in non-Riemannian geometry.

The characteristics of w-curvature tensors and their function in relativistic spacetimes and general relativity were the subject of early research ([10], [11], [33], [34] and [35]). Numerous scholars have examined generalized recurrent structures and their impact on curvature behavior within the context of Finsler geometry. Specifically, Al-Qashbari and associates used Cartan and Berwald covariant derivatives to construct a series of research addressing generalized recurrent, birecurrent, and trirecurrent Finsler spaces ([12], [13], [16] and [17]). In more recent publications, these analyses have been extended to higher-order curvature tensors, tensor decompositions, and generalized curvature relations in Finsler manifolds ([18], [19], [23] and [24]). Concurrent with these developments, recent research has emphasized the importance of curvature tensor analysis in understanding geometric invariants and structural properties of Finsler spaces ([30], [31], [32] and [35]).

Despite these advances, it is still unclear how Weyl-type curvature tensors relate geometrically to other curvature structures in Cartan Finsler spaces. Thus, studying the interaction between Weyl curvature tensors and P_1 -curvature tensors may lead to a deeper comprehension of curvature structures and recurrence properties in extended Finsler spaces ([26], [27], [28] and [29]).

Research Contributions

This paper's primary contributions can be summed up as follows:

1. Creating new geometric relations in Cartan Finsler spaces between the P_1 -curvature tensor and Weyl's projective curvature tensor.
2. Determining the sufficient and required requirements for generalized birecurrent and generalized recurrent structures.
3. Offering characterization theorems involving related tensors, including scalar curvature, torsion, deviation, and Ricci tensor.
4. Using higher-order Cartan covariant derivatives to extend recurrence theory in Finsler geometry.

II. Preliminaries

The fundamental definitions, notations, and geometric ideas needed to produce the findings in this study are presented in this part. We introduce the curvature tensors utilized in the analysis and provide a brief overview of the basic concepts of Cartan Finsler geometry [31].

Let $F^n(M, F)$ be an n -dimensional Finsler space with the Cartan connection. Finsler geometry's regularity and homogeneity requirements are met by the metric function $F(x, y)$, which is defined on the tangent bundle TM . ([32] and [36]).

The torsion tensor, deviation tensor, Ricci tensor, and scalar curvature are among the geometric tensors that naturally emerge within this paradigm. In particular, Weyl's projective curvature tensor plays an essential role in describing the projective invariance of the manifold ([15] and [25]).

P_1 -curvature tensor is another geometric object that captures specific curvature properties of the Finsler space. These tensors are related through several differential identities that involve covariant derivatives defined with respect to Cartan's connection ([21] and [22]).

In the following sections, these preliminary definitions will be used to derive recurrence relations between Weyl's curvature tensor and the P_1 -curvature tensor.

In this section, we provide the necessary conditions and definitions relevant to the purpose of this paper. Additionally, the two vectors y_i and y^i satisfy the following conditions:

$$\begin{aligned} & \text{a) } y_i = g_{ij} y^j, \quad \text{b) } y_i y^i = F^2 \\ & \text{and c) } \delta_j^k y^j = y^k. \end{aligned} \quad (2.1)$$

The quantities g_{ij} and g^{ij} are related as follows:

$$\begin{aligned} & \text{a) } g_{ij} g^{jk} = \delta_i^k = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases} \\ & \text{b) } g_{ij|_h} = 0, \quad \text{c) } g^{ij}|_h = 0, \quad \text{d) } g_{ir} \delta_j^i = g_{rj}, \\ & \text{and e) } g^{jk} \delta_k^i = g^{ji}. \end{aligned} \quad (2.2)$$

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j , successively, we get

$$\begin{aligned} X_{|k|_j}^i &= \partial_j(X_{|k}^i) - (X_{|r}^i) \Gamma_{kj}^{*r} + (X_{|k}^r) \Gamma_{rj}^{*i} \\ &- (\partial_j X_{|k}^i) \Gamma_{js}^{*i} y^s. \end{aligned} \quad (2.3)$$

The vector y^i and metric function F vanish identically under Cartan's covariant derivative

$$\text{a) } F_{|h} = 0, \quad \text{and b) } y^i|_h = 0. \quad (2.4)$$

The tensor W_{jkh}^i , the torsion tensor W_{jk}^i and the deviation tensor W_j^i are defined as follows:

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2\delta_j^i}{(n+1)} H_{[hk]} + \frac{2y^i}{(n+1)} \partial_j H_{[kh]} \\ &+ \frac{\delta_k^i}{(n^2-1)} (n H_{jh} + H_{hj} + y^r \partial_j H_{hr}) \\ &- \frac{\delta_h^i}{(n^2-1)} (n H_{jk} + H_{kj} + y^r \partial_j H_{kr}), \end{aligned} \quad (2.5)$$

$$\begin{aligned} W_{jk}^i &= H_{jk}^i + \frac{y^i}{(n+1)} H_{[jk]} \\ &+ 2 \left\{ \frac{\delta_j^i}{(n^2-1)} (n H_{k|} - y^r H_{k|r}) \right\}, \end{aligned} \quad (2.6)$$

$$\text{and } W_j^i = H_j^i - H \delta_j^i - \frac{1}{(n+1)} (\partial_r H_j^r - \partial_j H) y^i, \quad (2.7)$$

respectively.

Additionally, assuming that the tensor W_j^i satisfies the following identities

$$\begin{aligned} & \text{a) } W_k^i y^k = 0, \quad \text{b) } W_i^i = 0, \quad \text{c) } W_k^i y_i = 0, \\ & \text{d) } g_{ir} W_j^i = W_{rj}, \quad \text{e) } g^{jk} W_{jk} = W, \end{aligned}$$

and f) $W_{jk} y^k = 0$. (2.8)

We have the P_1 -curvature tensor P_{1jkh}^i , P_1 -torsion tensor P_{1jk}^i , P_1 -Ricci tensor P_{1jk} ,

P_1 -curvature vector P_{1k} , and P_1 -scalar curvature P_1 satisfying:

$$\begin{aligned} & \text{a) } P_{1jkh}^i y^j = P_{1kh}^i, \quad \text{b) } P_{1kh}^i y^k = P_{1h}^i, \\ & \text{c) } P_{1jki}^i = P_{1jk}, \quad \text{d) } P_{1ki}^i = P_{1k}, \\ & \text{e) } P_{1i}^i = P_1, \quad \text{and f) } g_{ir} P_{1jkh}^i = P_{1rjkh}. \end{aligned} \quad (2.9)$$

The Cartan third curvature tensor R_{jkh}^i , Ricci tensor R_{jk} , the vector H_k , and the scalar curvature H are defined as:

$$\begin{aligned} & \text{a) } R_{jk} y^j = H_k, \quad \text{b) } R_{jk} y^k = R_j, \quad \text{c) } R_i^i = R \\ & \text{and d) } H_k y^k = (n-1)H. \end{aligned} \quad (2.10)$$

Al-Qashbari and AL-ssallal [5], and Al-Qashbari, Haouse and AL-ssallal [7] introduced and studied generalization of Weyl's projective curvature tensor in Finsler spaces by: A Study on generalized

W -recurrent, Birecurrent, and Ricci Tensors using Cartan's and Berwald's first and second order derivatives in Finsler space which is characterized by the conditions:

$$\begin{aligned} W_{jkh|_m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}). \end{aligned} \quad (2.11)$$

A Finsler space F_n , in which the curvature tensor W_{jkh}^i satisfies the condition (2.11), is referred to as the generalized $W_{|h}$ -recurrent space and denoted by $G^{2nd} W_{|h}$ - RF_n .

By taking the covariant derivative of (2.11), with respect to x^l in the context of Cartan's connection and applying the relation (2.2b), we obtain:

$$\begin{aligned} W_{jkh|_m|_l}^i &= a_{ml} W_{jkh}^i + b_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} c_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m (W_{k|_l}^i g_{jh} - W_{h|_l}^i g_{jk}), \end{aligned} \quad (2.12)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ and $c_{ml} = (\lambda_m \gamma_l + \gamma_{m|l})$ are non-zero covariant tensors field of second order and γ_m is non-zero covariant vector of first order, respectively.

Definition 2.1. In Finsler space, which the Weyl's projective curvature tensor W_{jkh}^i satisfies the condition (2.12) is called the generalization generalized $W_{|h}$ -birecurrent space and the tensor will be called a generalization generalized h -birecurrent space. These space and tensor denote them briefly by $G^{2nd} W_{|h}$ - BRF_n and $G^{2nd} h$ - BR , respectively.

III. Relationship Between Weyl and P_1 -Curvature Tensors

In this section, we investigate the geometric relationship between Weyl's projective curvature tensor and the P_1 -curvature tensor in Cartan Finsler spaces. These tensors describe fundamental curvature properties of the manifold and play an important role in understanding its intrinsic geometric structure.

By applying Cartan's covariant derivative and using the identities introduced in the previous section, several relations between these curvature tensors can be derived. These relations provide conditions under which the curvature tensors exhibit generalized recurrent or generalized birecurrent structures.

The obtained identities reveal that the behavior of the P_1 -curvature tensor is closely connected to the recurrence properties of Weyl's curvature tensor. In particular, the analysis demonstrates that certain tensorial structures such as torsion tensors, deviation tensors, Ricci tensors, and scalar curvature are influenced by these recurrence relations.

These results provide a deeper geometric understanding of curvature interactions in Cartan Finsler spaces and establish the

theoretical foundation for the theorems presented in the next section.

Some properties of W_{jkh}^i curvature tensor was proposed by Al-Qashbari, Abdallah and Al-ssallal [8].

Definition 3.1. The space $G^{2nd}P_{1|h}$ is a generalized second-order P_1 -type Finsler space characterized by a recurrence condition along the horizontal distribution hhh. It extends classical recurrent Finsler spaces by incorporating higher-order curvature interactions. This structure allows the study of anisotropic flows and curvature tensor couplings in complex geometric frameworks.

For $(n = 4)$ a Riemannian space, Weyl defined the P_1 -curvature tensor $P_{1|jkh}^i$ often known as the Weyl P_1 -curvature tensor, as

$$W_{jkh}^i = P_{1|jkh}^i - \frac{1}{6}(\delta_h^i R_{jk} - g_{jh} R_k^i) + \frac{1}{3}(\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (3.1)$$

By taking the h - covariant derivative of (3.1), with respect to x^m , we obtain:

$$W_{jkh|m}^i = P_{1|jkh|m}^i - \frac{1}{6}(\delta_h^i R_{jk} - g_{jh} R_k^i)_{|m} + \frac{1}{3}(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|m}. \quad (3.2)$$

Using (2.2b), in the equation (3.2) can be written as

$$W_{jkh|m}^i = P_{1|jkh|m}^i - \frac{1}{6}(\delta_h^i R_{jk|m} - g_{jh} R_k^i) + \frac{1}{3}(\delta_k^i R_{jh|m} - g_{jk} R_h^i). \quad (3.3)$$

By substituting equations (2.11) and (3.1) into (3.3), we obtain:

$$P_{1|jkh|m}^i - \frac{1}{6}(\delta_h^i R_{jk|m} - g_{jh} R_k^i) + \frac{1}{3}(\delta_k^i R_{jh|m} - g_{jk} R_h^i) = \lambda_m P_{1|jkh}^i - \frac{1}{6} \lambda_m (\delta_h^i R_{jk} - g_{jh} R_k^i) + \frac{1}{3} \lambda_m (\delta_k^i R_{jh} - g_{jk} R_h^i) + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}). \quad (3.4)$$

The equation (3.4), can be expressed as:

$$P_{1|jkh|m}^i = \lambda_m P_{1|jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}) + \frac{1}{6} (\delta_h^i R_{jk|m} - g_{jh} R_k^i) - \frac{1}{3} (\delta_k^i R_{jh|m} - g_{jk} R_h^i) - \frac{1}{6} \lambda_m (\delta_h^i R_{jk} - g_{jh} R_k^i) + \frac{1}{3} \lambda_m (\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (3.5)$$

This demonstrates that

$$P_{1|jkh|m}^i = \lambda_m P_{1|jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}). \quad (3.6)$$

If and only if

$$(\delta_h^i R_{jk} - g_{jh} R_k^i)_{|m} = \lambda_m (\delta_h^i R_{jk} - g_{jh} R_k^i),$$

where $g_{jh|m} = 0$,

$$\text{and } (\delta_k^i R_{jh} - g_{jk} R_h^i)_{|m} = \lambda_m (\delta_k^i R_{jh} - g_{jk} R_h^i), \quad (3.7)$$

where $g_{jk|m} = 0$.

In conclusion the proof of theorem is completed, we can determine

Theorem 3.1. In the space $G^{2nd}P_{1|h} - RF_n$, the P_1 -curvature tensor $P_{1|jkh}^i$ represents a generalized recurrent Finsler space, provided that the condition (3.7) is satisfied.

By transvecting equation (3.5) with y^j and utilizing equations (2.9a), (2.4b), (2.1a) and (2.10a), we obtain the following result

$$P_{1|kh|m}^i = \lambda_m P_{1|kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k) + \frac{1}{4} \gamma_m (W_k^i y_h - W_h^i y_k) + \frac{1}{6} (\delta_h^i H_{k|m} - y_h R_k^i) - \frac{1}{3} (\delta_k^i H_{h|m} - y_k R_h^i) - \frac{1}{6} \lambda_m (\delta_h^i H_k - y_h R_k^i) + \frac{1}{3} \lambda_m (\delta_k^i H_h - y_k R_h^i). \quad (3.8)$$

This demonstrates that

$$P_{1|kh|m}^i = \lambda_m P_{1|kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k) + \frac{1}{4} \gamma_m (W_k^i y_h - W_h^i y_k). \quad (3.9)$$

If and only if

$$(\delta_h^i H_k - y_h R_k^i)_{|m} = \lambda_m (\delta_h^i H_k - y_h R_k^i),$$

where $y_{h|m} = 0$,

$$\text{and } (\delta_k^i H_h - y_k R_h^i)_{|m} = \lambda_m (\delta_k^i H_h - y_k R_h^i), \quad (3.10)$$

where $y_{k|m} = 0$.

Therefore, the proof of theorem is completed, we can say

Theorem 3.2. In the space $G^{2nd}P_{1|h} - RF_n$, the torsion tensor $P_{1|kh}^i$ represents a generalized recurrent Finsler space, provided that the condition (3.10) is satisfied.

By transvecting equation (3.8) with y^k and utilizing equations $(n = 4)$, (2.9b), (2.4b), (2.1b), (2.8a), (2.1c) and (2.10d), we obtain the following result

$$P_{1|h|m}^i = \lambda_m P_{1|h}^i + \mu_m (y^i y_h - \delta_h^i F^2) - \frac{1}{4} \gamma_m (W_h^i F^2) + \frac{1}{6} (3 \delta_h^i H_{|m} - y_h R_{k|m}^i y^k) - \frac{1}{3} (y^i H_{h|m} - F^2 R_{h|m}^i) - \frac{1}{6} \lambda_m (3 \delta_h^i H - y_h R_k^i y^k) + \frac{1}{3} \lambda_m (y^i H_h - F^2 R_h^i). \quad (3.11)$$

This demonstrates that

$$P_{1|h|m}^i = \lambda_m P_{1|h}^i + \mu_m (y^i y_h - \delta_h^i F^2) - \frac{1}{4} \gamma_m (W_h^i F^2). \quad (3.12)$$

If and only if

$$(3 \delta_h^i H - y_h R_k^i y^k)_{|m} = \lambda_m (3 \delta_h^i H - y_h R_k^i y^k),$$

where $(y_h y^k)_{|m} = 0$,

$$\text{and } (y^i H_h - F^2 R_h^i)_{|m} = \lambda_m (y^i H_h - F^2 R_h^i), \quad (3.13)$$

where $(y^i - F^2)_{|m} = 0$.

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. In the space $G^{2nd}P_{1|h} - RF_n$, the P_1 - deviation tensor $P_{1|h}^i$ represents a generalized recurrent Finsler space if the tensors $(3 \delta_h^i H - y_h R_k^i y^k)$ and $(y^i H_h - F^2 R_h^i)$ are generalized recurrent Finsler spaces.

By contracting the indices i and h in equations (3.5), (3.8), and (3.11), and utilizing equations $(n = 4)$, (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), in conjunction with (2.9c), (2.9d), and (2.9e), we obtain the following result:

$$P_{1|jk|m} = \lambda_m P_{1|jk} + (1 - n) \mu_m g_{jk} + \frac{1}{4} \gamma_m W_{jk} + \frac{1}{6} (n - 1) R_{jk|m} - \frac{1}{3} (R_{jk|m} - g_{jk} R_{|m}) - \frac{1}{6} (n - 1) \lambda_m R_{jk} + \frac{1}{3} \lambda_m (R_{jk} - g_{jk} R). \quad (3.14)$$

This demonstrates that

$$P_{1|jk|m} = \lambda_m P_{1|jk} + (1 - n) \mu_m g_{jk} + \frac{1}{4} \gamma_m W_{jk}. \quad (3.15)$$

If and only if

$$R_{jk|m} = \lambda_m R_{jk},$$

$$\text{and } (R_{jk} - g_{jk} R)_{|m} = \lambda_m (R_{jk} - g_{jk} R), \quad (3.16)$$

where $g_{jk|m} = 0$.

and $P_{1_k|m} = \lambda_m P_{1_k} + (1-n)\mu_m y_k$
 $+ \frac{1}{6}(nH_{k|m} - y_i R_{k|m}^i) - \frac{1}{3}(H_{k|m} - y_k R_{|m})$
 $- \frac{1}{6}\lambda_m(nH_k - y_i R_k^i) + \frac{1}{3}\lambda_m(H_k - y_k R)$.

This demonstrates that

$$P_{1_k|m} = \lambda_m P_{1_k} + (1-n)\mu_m y_k. \quad (3.18)$$

If and only if

$$(nH_k - y_i R_k^i)_{|m} = \frac{1}{2}\lambda_m(nH_k - y_i R_k^i),$$

where $y_{i|m} = 0$

and $(H_k - y_k R)_{|m} = \lambda_m(H_k - y_k R)$,

where $y_{k|m} = 0$.

In the last

$$P_{1|m} = \lambda_m P_1 + (1-n)\mu_m F^2$$

$$+ \frac{1}{6}(3nH_{|m} - y_i R_{k|m}^i y^k) - \frac{1}{3}(3H_{|m} - F^2 R_{|m})$$

$$- \frac{1}{6}\lambda_m(3nH - y_i R_k^i y^k) + \frac{1}{3}\lambda_m(3H - F^2 R). \quad (3.20)$$

This demonstrates that

$$P_{1|m} = \lambda_m P_1 + (1-n)\mu_m F^2. \quad (3.21)$$

If and only if

$$(3nH - y_i R_k^i y^k)_{|m} = \lambda_m(3nH - y_i R_k^i y^k),$$

where $(y_i y^k)_{|m} = 0$,

and $(3H - F^2 R)_{|m} = \lambda_m(3H - F^2 R)$,

where $F_{|m}^2 = 0$.

In conclusion the proof of theorem is completed, we can say

Theorem 3.4. In the space $G^{2nd}P_{1|_h} - RF_n$, the P_1 -Ricci tensor P_{1_jk} , the P_1 -vector P_{1_k} and the P_1 -scalar P_1 are defined in equations (3.15), (3.18), and (3.21), respectively, if and only if the conditions in equations (3.16), (3.19), and (3.22) are satisfied.

By transvecting equation (3.5) with g_{ir} and utilizing equations (2.2d), (2.2b), (2.8d), and (2.9f), we obtain the following result

$$P_{1_rjkh|m} = \lambda_m P_{1_rjkh} + \mu_m(g_{rk}g_{jh} - g_{rh}g_{jk})$$

$$+ \frac{1}{4}\gamma_m(W_{rk}g_{jh} - W_{rh}g_{jk})$$

$$+ \frac{1}{6}(g_{rh}R_{jk|m|l} - g_{jh}R_{rk|m|l})$$

$$- \frac{1}{3}(g_{rk}R_{jh|m|l} - g_{jk}R_{rh|m|l}) - \frac{1}{6}\lambda_m(g_{rh}R_{jk} - g_{jh}R_{rk})$$

$$+ \frac{1}{3}\lambda_m(g_{rk}R_{jh} - g_{jk}R_{rh}). \quad (3.23)$$

This demonstrates that

$$P_{1_rjkh|m} = \lambda_m P_{1_rjkh} + \mu_m(g_{rk}g_{jh} - g_{rh}g_{jk})$$

$$+ \frac{1}{4}\gamma_m(W_{rk}g_{jh} - W_{rh}g_{jk}). \quad (3.24)$$

If and only if

$$(g_{rh}R_{jk} - g_{jh}R_{rk})_{|m} = \lambda_m(g_{rh}R_{jk} - g_{jh}R_{rk}),$$

where $g_{jh|m} = 0$,

and $(g_{rk}R_{jh} - g_{jk}R_{rh})_{|m} = \lambda_m(g_{rk}R_{jh} - g_{jk}R_{rh})$. (3.25)

Therefore, the proof of theorem is completed, we can say

Theorem 3.5. In the space $G^{2nd}P_{1|_h} - RF_n$, the associate tensor P_{1_rjkh} represents a generalized recurrent Finsler space if the condition in equation (3.25) is satisfied.

We introduce a new class of Finsler spaces, namely, generalized- $P_{1|_h}$ -birecurrent spaces. These spaces extend the concept of birecurrence to a broader context and exhibit interesting geometric properties. In this study, we analyze the

curvature tensor of these spaces and establish several characterization theorems. Specifically, we define $|m|l$ as the covariant derivative of second order.

By taking the h-covariant derivative of equation (3.1) with respect to x^m and x^l , respectively, we obtain the following result.

$$W_{jkh|m|l}^i = P_{1_jkh|m|l}^i - \frac{1}{6}(\delta_h^i R_{jk} - g_{jh} R_k^i)_{|m|l}$$

$$+ \frac{1}{3}(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|m|l}. \quad (3.26)$$

Using equation (2.2b), equation (3.26) can be rewritten as follows

$$W_{jkh|m|l}^i = P_{1_jkh|m|l}^i - \frac{1}{6}(\delta_h^i R_{jk|m|l} - g_{jh} R_{k|m|l}^i)$$

$$+ \frac{1}{3}(\delta_k^i R_{jh|m|l} - g_{jk} R_{h|m|l}^i). \quad (3.27)$$

Similarly, by applying equations (2.12) and (3.1) in (3.27), we obtain the result

$$P_{1_jkh|m|l}^i - \frac{1}{6}(\delta_h^i R_{jk|m|l} - g_{jh} R_{k|m|l}^i)$$

$$+ \frac{1}{3}(\delta_k^i R_{jh|m|l} - g_{jk} R_{h|m|l}^i) = a_{ml} P_{1_jkh}^i$$

$$- \frac{1}{6}a_{ml}(\delta_h^i R_{jk} - g_{jh} R_k^i) + \frac{1}{3}a_{ml}(\delta_k^i R_{jh} - g_{jk} R_h^i)$$

$$+ b_{ml}(\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \frac{1}{4}c_{ml}(W_k^i g_{jh} - W_h^i g_{jk})$$

$$+ \frac{1}{4}\gamma_m(W_{k|l}^i g_{jh} - W_{h|l}^i g_{jk}).$$

Alternatively, this can be expressed as:

$$P_{1_jkh|m|l}^i = a_{ml} P_{1_jkh}^i + b_{ml}(\delta_k^i g_{jh} - \delta_h^i g_{jk})$$

$$+ \frac{1}{4}c_{ml}(W_k^i g_{jh} - W_h^i g_{jk})$$

$$+ \frac{1}{4}\gamma_m(W_{k|l}^i g_{jh} - W_{h|l}^i g_{jk}) + \frac{1}{6}(\delta_h^i R_{jk|m|l} - g_{jh} R_{k|m|l}^i)$$

$$- \frac{1}{3}(\delta_k^i R_{jh|m|l} - g_{jk} R_{h|m|l}^i) + \frac{1}{6}a_{ml}(\delta_h^i R_{jk} - g_{jh} R_k^i)$$

$$+ \frac{1}{3}a_{ml}(\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (3.28)$$

This demonstrates that

$$P_{1_jkh|m|l}^i = a_{ml} P_{1_jkh}^i + b_{ml}(\delta_k^i g_{jh} - \delta_h^i g_{jk})$$

$$+ \frac{1}{4}c_{ml}(W_k^i g_{jh} - W_h^i g_{jk})$$

$$+ \frac{1}{4}\gamma_m(W_{k|l}^i g_{jh} - W_{h|l}^i g_{jk}). \quad (3.29)$$

If and only if

$$(\delta_h^i R_{jk} - g_{jh} R_k^i)_{|m|l} = a_{ml}(\delta_h^i R_{jk} - g_{jh} R_k^i),$$

where $g_{jh|m|l} = 0$,

and $(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|m|l} = a_{ml}(\delta_k^i R_{jh} - g_{jk} R_h^i)$,

where $g_{jk|m|l} = 0$.

$$(3.30)$$

In conclusion the proof of theorem is completed, we can determine

Theorem 3.6. In the space $G^{2nd}P_{1|_h} - BRF_n$, the P_1 -curvature tensor $P_{1_jkh}^i$ defines a generalized birecurrent Finsler space if and only if the condition in equation (3.30) is satisfied. By transvecting condition (3.28) with y^j , and utilizing equations (2.9a), (2.4b), (2.1a) and (2.10a), we obtain the following result.

$$P_{1_kh|m|l}^i = a_{ml} P_{1_kh}^i + b_{ml}(\delta_k^i y_h - \delta_h^i y_k)$$

$$+ \frac{1}{4}c_{ml}(W_k^i y_h - W_h^i y_k) + \frac{1}{4}\gamma_m(W_{k|l}^i y_h - W_{h|l}^i y_k)$$

$$+ \frac{1}{6}(\delta_h^i H_{k|m|l} - y_h R_{k|m|l}^i) - \frac{1}{3}(\delta_k^i H_{h|m|l} - y_k R_{h|m|l}^i)$$

$$- \frac{1}{6}a_{ml}(\delta_h^i H_k - y_h R_k^i) + \frac{1}{3}a_{ml}(\delta_k^i H_h - y_k R_h^i). \quad (3.31)$$

This demonstrates that

$$P_{1_kh|m|l}^i = a_{ml} P_{1_kh}^i + b_{ml}(\delta_k^i y_h - \delta_h^i y_k)$$

$$\begin{aligned}
 & + \frac{1}{4} c_{ml} (W_k^i y_h - W_h^i y_k) \\
 & + \frac{1}{4} \gamma_m (W_{k|l}^i y_h - W_{h|l}^i y_k). \quad (3.32)
 \end{aligned}$$

If and only if

$$(\delta_h^i H_k - y_h R_k^i)_{|m|l} = a_{ml} (\delta_h^i H_k - y_h R_k^i),$$

where $y_{h|m|l} = 0$,

$$\text{and } (\delta_k^i H_h - y_k R_h^i)_{|m|l} = a_{ml} (\delta_k^i H_h - y_k R_h^i), \quad (3.33)$$

where $y_{k|m|l} = 0$.

Therefore, the proof of theorem is completed, we can say **Theorem 3.7.** In the space $G^{2nd}P_{1|h} - BRF_n$, the h-covariant derivative of second-order for the torsion tensor P_{1kh}^i defines a generalized birecurrent Finsler space if and only if the condition in equation (3.33) is satisfied.

By transvecting condition (3.31) with y^k , and applying ($n = 4$), along with equations (2.9b), (2.4b), (2.1b), (2.8a), (2.1c), and (2.10d), we obtain the following result.

$$\begin{aligned}
 P_{1h|m|l}^i & = a_{ml} P_{1h}^i + b_{ml} (y^i y_h - \delta_h^i F^2) \\
 & - \frac{1}{4} c_{ml} (W_h^i F^2) - \frac{1}{4} \gamma_m W_{h|l}^i F^2 \\
 & + \frac{1}{6} (3\delta_h^i H_{|m|l} - y_h R_{k|m|l}^i y^k) \\
 & - \frac{1}{3} (y^i H_{h|m|l} - F^2 R_{h|m|l}^i) - \frac{1}{6} a_{ml} (3\delta_h^i H - y_h R_k^i y^k) \\
 & + \frac{1}{3} a_{ml} (y^i H_h - F^2 R_h^i). \quad (3.34)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 P_{1h|m|l}^i & = a_{ml} P_{1h}^i + b_{ml} (y^i y_h - \delta_h^i F^2) \\
 & - \frac{1}{4} c_{ml} (W_h^i F^2) - \frac{1}{4} \gamma_m W_{h|l}^i F^2. \quad (3.35)
 \end{aligned}$$

If and only if

$$(3\delta_h^i H - y_h R_k^i y^k)_{|m|l} = a_{ml} (3\delta_h^i H - y_h R_k^i y^k),$$

where $(y_h y^k)_{|m|l} = 0$,

$$\text{and } (y^i H_h - F^2 R_h^i)_{|m|l} = a_{ml} (y^i H_h - F^2 R_h^i), \quad (3.36)$$

where $(y^i - F^2)_{|m|l} = 0$.

Therefore, the proof of theorem is completed, we can say **Theorem 3.8.** In the space $G^{2nd}P_{1|h} - BRF_n$, the P_1 -deviation tensor P_{1h}^i represents a generalized birecurrent Finsler space if the tensors $(3\delta_h^i H - y_h R_k^i y^k)$ and $(y^i H_h - F^2 R_h^i)$ are generalized birecurrent Finsler spaces.

By contracting the indices i and h in equations (3.28), (3.31), and (3.34), and utilizing equations ($n = 4$), (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), along with the relations in equations (2.9c), (2.9d), and (2.9e), we obtain the following result.

$$\begin{aligned}
 P_{1jk|m|l} & = a_{ml} P_{1jk} + (1-n)b_{ml} g_{jk} + \frac{1}{4} c_{ml} W_{jk} \\
 & + \frac{1}{4} \gamma_m W_{jk|l} + \frac{1}{6} (n-1) R_{jk|m|l} \\
 & - \frac{1}{3} (R_{jk|m|l} - g_{jk} R_{|m|l}) - \frac{1}{6} (n-1) a_{ml} R_{jk} \\
 & + \frac{1}{6} a_{ml} (R_{jk} - g_{jk} R). \quad (3.37)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 P_{1jk|m|l} & = a_{ml} P_{1jk} + (1-n)b_{ml} g_{jk} \\
 & + \frac{1}{4} c_{ml} W_{jk} + \frac{1}{4} \gamma_m W_{jk|l}. \quad (3.38)
 \end{aligned}$$

If and only if

$$\begin{aligned}
 R_{jk|m|l} & = a_{ml} R_{jk}, \\
 (R_{jk} - g_{jk} R)_{|m|l} & = a_{ml} (R_{jk} - g_{jk} R), \quad (3.39)
 \end{aligned}$$

where $g_{jk|m|l} = 0$.

And

$$\begin{aligned}
 P_{1k|m|l} & = a_{ml} P_{1k} + (1-n)b_{ml} y_k \\
 & + \frac{1}{6} (nH_{k|m|l} - y_i R_{k|m|l}^i) - \frac{1}{3} (H_{k|m|l} - y_k R_{|m|l}) \\
 & - \frac{1}{6} a_{ml} (nH_k - y_i R_k^i) + \frac{1}{3} a_{ml} (H_k - y_k R). \quad (3.40)
 \end{aligned}$$

This demonstrates that

$$P_{1k|m|l} = a_{ml} P_{1k} + (1-n)b_{ml} y_k. \quad (3.41)$$

If and only if

$$(nH_k - y_i R_k^i)_{|m|l} = a_{ml} (nH_k - y_i R_k^i),$$

where $y_{i|m|l} = 0$,

$$\text{and } (H_k - y_k R)_{|m|l} = a_{ml} (H_k - y_k R), \quad (3.42)$$

where $y_{k|m|l} = 0$.

In the last

$$\begin{aligned}
 P_{1|m|l} & = a_{ml} P_1 + (1-n)b_{ml} F^2 \\
 & - \frac{1}{2} (3nH_{|m|l} - y_i R_{k|m|l}^i y^k) - \frac{1}{6} (3H_{|m|l} - F^2 R_{|m|l}) \\
 & + \frac{1}{2} a_{ml} (3nH - y_i R_k^i y^k) + \frac{1}{6} a_{ml} (3H - F^2 R). \quad (3.43)
 \end{aligned}$$

This demonstrates that

$$P_{1|m|l} = a_{ml} P_1 + (1-n)b_{ml} F^2. \quad (3.44)$$

If and only if

$$(3nH - y_i R_k^i y^k)_{|m|l} = a_{ml} (3nH - y_i R_k^i y^k),$$

where $(y_i y^k)_{|m|l} = 0$,

$$\text{and } (3H - F^2 R)_{|m|l} = a_{ml} (3H - F^2 R), \quad (3.45)$$

where $F_{|m|l}^2 = 0$.

Therefore, the proof of theorem is completed, we can say

Theorem 3.9. In the space $G^{2nd}P_{1|h} - BRF_n$, the P_1 -Ricci tensor P_{1jk} , the P_1 -vector P_{1k} and the P_1 -scalar P_1 are defined in equations (3.38), (3.41), and (3.44), respectively, provided that the conditions (3.39), (3.42), and (3.45) are satisfied.

By transvecting equation (3.28) with g_{ir} and applying equations (2.2d), (2.2b), (2.8d), and (2.9f), we obtain the following result:

$$\begin{aligned}
 P_{1rjkh|m|l} & = a_{ml} P_{1rjkh} + b_{ml} (g_{rk} g_{jh} - g_{rh} g_{jk}) \\
 & + \frac{1}{4} c_{ml} (W_{rk} g_{jh} - W_{rh} g_{jk}) + \frac{1}{4} \gamma_m (W_{rk|l} g_{jh} - W_{rh|l} g_{jk}) \\
 & + \frac{1}{6} (g_{rh} R_{jk|m|l} - g_{jh} R_{rk|m|l}) \\
 & - \frac{1}{6} a_{ml} (g_{rh} R_{jk} - g_{jh} R_{rk}) - \frac{1}{3} (g_{rk} R_{jh|m|l} - g_{jk} R_{rh|m|l}) \\
 & + \frac{1}{3} a_{ml} (g_{rk} R_{jh} - g_{jk} R_{rh}). \quad (3.46)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 P_{1rjkh|m|l} & = a_{ml} P_{1rjkh} + b_{ml} (g_{rk} g_{jh} - g_{rh} g_{jk}) \\
 & + \frac{1}{4} c_{ml} (W_{rk} g_{jh} - W_{rh} g_{jk}) \\
 & + \frac{1}{4} \gamma_m (W_{rk|l} g_{jh} - W_{rh|l} g_{jk}). \quad (3.47)
 \end{aligned}$$

If and only if

$$(g_{rh} R_{jk} - g_{jh} R_{rk})_{|m|l} = a_{ml} (g_{rh} R_{jk} - g_{jh} R_{rk}),$$

where $g_{jh|m|l} = 0$, and

$$(g_{rk} R_{jh} - g_{jk} R_{rh})_{|m|l} = a_{ml} (g_{rk} R_{jh} - g_{jk} R_{rh}). \quad (3.48)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.10. In the space $G^{2nd}P_{1|h} - BRF_n$, the associate tensor P_{1rjkh} (P_1 -curvature tensor P_{1rjkh}^i) characterizes a generalized birecurrent Finsler space, provided that condition (3.48) is satisfied.

IV. Conclusions

This paper investigated the geometric relations between Weyl's projective curvature tensor and the P_1 -curvature tensor in Cartan Finsler spaces. By employing higher-order Cartan covariant derivatives, several new tensorial identities and recurrence relations were derived. The obtained results provide a unified characterization of several geometric quantities including torsion tensors, deviation tensors, Ricci tensors, and scalar curvature. These findings extend existing results on recurrent structures in Finsler geometry and highlight deeper interactions between curvature tensors in generalized recurrent spaces. All things considered, the research advances differential geometry theory and offers fresh perspectives on the curvature tensor structure in Cartan Finsler manifolds.

V. Future Work / Recommendations

These results may be extended to additional types of curvature tensors in generalized Finsler spaces in future research. Additionally, more research might look into the function of higher-order curvature relations in more general geometric frameworks and how they might relate to geometric analysis.

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