Fourth-Order P^h-Recurrent Structures in Affinely Connected Finsler Spaces with Perspectives on Nonlinear Stability

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Abstract— In the context of affinely connected Finsler spaces, the theoretical underpinnings of higher-order recurrent geometric forms are examined in this research. With a particular focus on Cartan's second curvature tensor P^i_{jkh} , we examine the necessary and sufficient conditions for the existence of generalized P^{h} -four-recurrent spaces. Several theorems governing the behavior of linked curvature and torsion tensors are derived by using the P^{h}-covariant derivative of the fourth order. In addition to the mathematical derivation, the article talks about how these higher-order structures can be used in nonlinear stability analysis. The "curvature" and "recurrence" characteristics of these spaces, where non-linear fluctuations and systemic persistence are common, offer a strong mathematical analogy for simulating the dynamics of fintech-driven financial stability and the course of economic recovery.
This paper establishes a mathematical analogy between the directional curvature of Finsler manifolds and the non-linear fluctuations in fintech-driven financial systems.

Key words: *Generalized P^h-Four-Recurrent Space; Nonlinear Stability Analysis; h-Covariant Derivative; Systemic Risk Modeling; Geometric Economics; Higher-Order Recurrence.*

LIST OF SYMBOLS

1) h-C.D.O.: h-covariant derivative of order.
2) P^i_{jkh} : Cartan's second curvature tensor.
3) g_{jk} : Finsler metric tensor.
4) C^n_{mr} : Cartan torsion tensor.

I. Introduction

The study of recurrent spaces has long been a cornerstone of differential geometry, offering profound insights into the intrinsic properties of manifolds. In recent years, the evolution of Finsler geometry, a generalization of Riemannian geometry, has provided a more flexible framework for describing systems where the directional argument plays a critical role. Among these, affinely connected Finsler spaces stand out due to their rich geometric structure and their ability to characterize complex directional dependencies.
This research focuses on a novel generalization: Generalized P^{h} -Four-Recurrent Spaces. While traditional recurrence deals with first or second-order derivatives, higher-order recurrence (specifically the fourth order) allows for a more granular analysis of how geometric properties persist or dissipate across a manifold.
The motivation for this work is twofold. First, establish a rigorous mathematical framework for P^{h} -G-FR F_{n} (Generalized Four-Recurrent Finsler spaces) and prove the essential theorems regarding their curvature tensors. Second, to bridge the gap between abstract geometry and economic modeling. In the context of modern financial ecosystems, banking stability is no longer a linear phenomenon; it is "Fintech-driven," characterized by high-frequency shifts and non-linear interdependencies. By applying the principles of higher-order recurrence and stability analysis, we can better understand the "curvature" of economic cycles and the geometric pathways leading to sustainable economic recovery.
Although recurrent structures of first, second, and third order have been extensively studied, the fourth-order P^h -recurrence has not been systematically developed for affinely connected Finsler spaces. This paper fills this gap by establishing complete necessary and sufficient conditions for generalized fourth-order recurrence.

Literature Review

Rund (1959) laid the groundwork for Finsler geometry by offering a rigorous framework for the differential geometry of Finsler spaces. Higher-order generalizations and recurrence features in these spaces have received a lot of scholarly interest in recent years. Recurrent Finsler structures and the decomposition of curvature tensors have been thoroughly studied by Al-Qashbari et al. (2024; 2025), with a particular emphasis on Weyl's and conharmonic curvature tensors employing Berwald's and Cartan's higher-order derivatives. Additionally, Al-Qashbari's work (2017; 2019; 2020) has expanded the study of trirecurrent and birecurrent qualities by introducing new identities for generalized curvature tensors and analyzing their behavior under different covariant derivatives.
These ideas have been expanded upon by recent contributions in 2025, such as the use of Lie derivatives in the analysis of M-projective curvature tensors and generalized n^{th}-order recurrent tensor fields (Al-Qashbari et al., 2025). These developments illustrate the rising complexity of curvature structures and their breakdown analysis on modern Finsler manifolds.

The fast integration of financial technology (FinTech) has altered the global banking scene, triggering a rise in academic research regarding its impact on financial stability and risk management. Recent studies (2025-2026) shift from linear models to advanced geometric frameworks, with Akcora et al. (2026) and Dergi (2025) utilizing topological analysis and Ricci flow to map structural breaks and market manifolds. Furthermore, Tayebi and Moghaddam (2025) demonstrate the efficacy of Finsler H-curvature in nonlinear dynamics, while Wang and Jadbabaie (2025) provide a geometric perspective on FinTech-driven systemic risk. Together, these works establish a rigorous mathematical foundation for assessing banking stability and resilience in modern, digitally transformed economies.

The following formula defines a P^h-recurrent space:

(1.1) $P^i_{jkh|\ell} = \lambda_\ell P^i_{jkh} + \mu_\ell(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $P^i_{jkh} \neq 0$,

where $|\ell$ represents the covariant derivative, The recurrence vector field, or λ_ℓ , is a non-trivial covariant vector field, and P^i_{jkh} is a tensor.

We then presented the notion of P^h-birecurrent spaces, which are defined as follows:

(1.2) $P^i_{jkh|\ell|m} = a_{\ell m} P^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $P^i_{jkh} \neq 0$.

The birecurrence tensor fields, denoted by $a_{\ell m}$ and $b_{\ell m}$, are second-order non- trivial covariant tensor fields.

The vectors y^j and y_j have an identical vanishing covariant derivative, meaning that the vectors y_j and δ^i_k fulfill the following. The tensor g_{kh} and the associated tensor g^{kh} are covariant constants.

(1.3) a) $g_{kh|r} = 0$ and b) $g^{kh}_{|r} = 0$.

(1.4) $g_{kr}\, g^{rh} = \delta^h_k = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{if } k \neq h \end{cases}$.

(1.5) a) $y^j_{|m} = 0$ and b) $y_{j|m} = 0$.

(1.6) a) $y_j\, y^j = F^2$ and b) $\dot{\partial}_m y_n = \dot{\partial}_n y_m = g_{mn}$.

(1.7) a) $\delta^i_j\, y^j = y^i$ and b) $\delta^j_m\, y_j = y_m$

(1.8) a) $\delta^m_n\, g^{nj} = g^{mj}$ and b) $\delta^m_j\, \delta^j_n = \delta^m_n$

(1.9) a) $\delta^m_n\, g_{jm} = g_{jn}$ and b) $g_{jm}\, y^j = y_m$.

This tensor satisfies the identities using Euler's on homogeneous characteristics.

(1.10) a) $C_{rmn}\, y^r = C_{mrn}\, y^r = C_{mnr}\, y^r = 0$ and b) $C^n_{rm}\, y^r = C^n_{mr}\, y^r = 0$.

The hv-curvature tensor, which is positively homogeneous of degree zero in the directional argument, is described by Cartan's second curvature tensor.

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm}\, P^m_{kh} - C^i_{jh|k} ,$$

or comparable by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k} ,$$

or

$$P^i_{jkh} = C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk}\, P^i_{rh} - P^r_{jh}\, C^i_{rk} .$$

The v(hv)-torsion tensor P^i_{kh} , the hv-curvature tensor P^i_{jkh} , its associated curvature tensor P_{ijkh} , and the P-Ricci tensor P_{jk} and tensor P^i_k satisfy.

(1.11) a) $P^i_{jkh}\, y^j = P^i_{kh}$, b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$, c) $P^i_{jkh} = P_{pjkh}\, g^{ip}$,
d) $P^i_{jki} = P_{jk}$ and e) $P^i_{ki} = P_k$, f) $P^i_{hk}\, y^h = P^i_k$ and g) $P^i_{jkh} = \dot{\partial}_j P^i_{kh}$.

The scalar curvature H can be found using

(1.12) $H^i_i = (1 - n)\, H$.

The quantities that make up the components of tensors, H^i_{jkh} and H^i_{kh} , are defined as follows:

(1.13) a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj} G^r_k - \partial_j G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$, and
b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - \partial_k G^i_h - G^r_h\, C^i_{rk}$.

Additionally, they are connected by

(1.14) a) $H^i_{jmn}\, y^j = H^i_{mn}$, b) $g_{mn}\, H^m_{jkh} = H_{jnkh}$ and c) $H^m_{jn} = \dot{\partial}_j H^m_n$.

For every given vector X^i, the operation of differentiation w. r. t. x^k commutes with partial differentiation with respect to y^j .

(1.15) $\dot{\partial}_j \left(X^n_{|m}\right) = \left(\dot{\partial}_j X^n\right)_{|m} + X^r\left(\dot{\partial}_j \Gamma^{*n}_{rm}\right) - \left(\dot{\partial}_r X^n\right) P^r_{jm}$.

The homogeneous functions connected by Euler's theorem

(1.16) a) $H^n_{jm}\, y^j = -H^n_{mj}\, y^j = H^n_m$ and b) $g_{np}\, H^n_{jm} = H_{jpm}$.

2. Essential and Enough Generalized Conditions to Determine P^h-Recurrent

The necessary and sufficient requirements for detecting P-recurrent patterns in a generalized setting are examined in this study. By defining these requirements, we hope to offer a strict framework for identifying and evaluating P-recurrent phenomena across a range of fields.

Mathematicians have long explored generalized four-recurrent affinely connected spaces, a unique kind of manifold. These areas are employed in many different applications and have several intriguing characteristics. The necessary and sufficient conditions for a space to be a generalized four recurrent affinely connected space will be covered in this work. We will also talk about a few uses for these places. P^i_{jkh} , the second Cartan's curvature tensor met the requirements for generalized recurrence.

(2.1) $P^i_{jkh|\ell} = \lambda_\ell\, P^i_{jkh} + \mu_\ell(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $P^i_{jkh} \neq 0$,

The space is known as a generalized P^h-recurrent space, where $|l$ is an h-C.D.O. one w. r. t. x^l and λ_ℓ , μ_ℓ are non-zero covariant vector fields.

Additionally, the generalized birecurrence requirement was satisfied by the curvature tensor P^i_{jkh}.

(2.2) $P^i_{jkh|\ell|m} = a_{\ell m}\, P^i_{jkh} + b_{\ell m}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $P^i_{jkh} \neq 0$,

where $|l|m$ is an h-C.D.O. two w. r. t. x^l and x^m, and α_{lm} and β_{lm} are non-trivial covariant vector fields. The space is referred to as a generalized P^h-birecurrent space.

Applying (1.5a) after covariantly differentiating (2.2), with respect to x^n produces

(2.3) $P^i_{jkh|\ell|m|n} = \alpha_{\ell mn} P^i_{jkh} + \beta_{\ell mn}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $P^i_{jkh} \neq 0$.

In this case, $\alpha_{\ell mn} = a_{\ell|mn} + a_{\ell m}\lambda_n$ and $\beta_{\ell mn} = a_{\ell m}\,\mu_n + b_{\ell|m|n}$ are non-trivial covariant tensor fields of order three, and $|\ell|m|n$ is a successful h-C.D.O. three w. r. t. x^ℓ , x^m and x^n .

Using (1.5a) and covariant differentiation of (2.3), with regard to x^r, we obtain

(2.4) $$P^i_{jkh|\ell|m|n|r} = \alpha_{\ell mnr}P^i_{jkh} + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \;,\; P^i_{jkh} \neq 0 \;.$$

In this instance, the non-trivial covariant tensor fields of rank four are $\alpha_{\ell mnr} = a_{\ell|mnr} + a_{\ell mr}\lambda_n + a_{\ell m}\lambda_{nr}$ and $\beta_{\ell mnr} = a_{\ell mr}\,\mu_{nm} + a_{\ell m}\,\mu_{nrm} + b_{\ell|m|n|r}$. The fourth-order h-covariant derivative with regard to x^ℓ, x^m, x^n and x^r, in that sequence.

The space and tensor that satisfy (2.4) are referred to as the P^h-generalized four recurrent spaces.

In short, we will refer to them as P^h-G-FRF$_n$.

Outcome 2.1. All generalized P^h-Fourecurrent spaces are also generalized P^h -Trirecurrent spaces.

Using (1.3a), (1.9a), and (1.11b) to transvect (2.4) by g_{is} , we obtain

(2.5) $$P_{jskh|\ell|m|n|r} = \alpha_{\ell mnr}P_{jskh} + \beta_{\ell mnr}(g_{ks}g_{jh} - g_{hs}g_{jk}) \;,\; P_{jrkh} \neq 0 \;.$$

On the other hand, condition (2.4) is obtained by transvecting (2.5) by g^{is} , using (1.3a), (1.4), and (1.11c). Consequently, the following theorem can be stated.

Theorem 2.1. The fourth-order h-covariant derivative of the associated curvature tensor P_{ijkh} of the curvature tensor P^i_{jkh} in P^h-G-FRF$_n$, is provided by (2.5).

Using (1.5a), (1.11a), and (1.14a) to transvect (2.4) by y^j, we obtain

(2.6) $$P^i_{kh|\ell|m|n|r} = \alpha_{\ell mnr}P^i_{kh} + \beta_{\ell mnr}(\delta^i_k\, y_h - \delta^i_h\, y_k) \;.$$

Using (1.5a), (1.11f), (1.7a), and (1.17a) to transvect (2.6) by y^k, we obtain

(2.7) $$P^i_{h|\ell|m|n|r} = \alpha_{\ell mnr}P^i_h + \beta_{\ell mnr}(y^i y_h - \delta^i_h\, F^2) \;.$$

As a result, Theorem 2.2 is true.

Theorem 2.2. The h-covariant derivative of 4th order for the deviation tensor P^i_h and the h(v)-torsion tensor P^i_{kh} in P^h-G-FRF$_n$ are given by (2.6) and (2.7), respectively.

Using (1.6b) and (1.11g) to differentiate (2.6) with regard to y^j , we obtain

(2.8) $$\dot\partial_j\left(P^i_{kh|\ell|m|n|r}\right) = (\dot\partial_j\alpha_{\ell mnr})\,P^i_{kh} + \alpha_{\ell mnr}\,P^i_{jkh} + (\dot\partial_j\beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \;.$$

Using the equ. of (1.15) for $\left(P^i_{kh|\ell|m|n}\right)$ in (2.8), we obtain

(2.9) $$\dot\partial_j(P^i_{kh|\ell|m|n})_{|r} + P^r_{kh|\ell|m|n}\dot\partial_j\Gamma^{*i}_{sr} - P^i_{sh|\ell|m|n}\dot\partial_j\Gamma^{*t}_{kr} - P^i_{ks|\ell|m|n}\dot\partial_j\Gamma^{*t}_{hr} - P^i_{kh|s|m|n}\dot\partial_j\Gamma^{*t}_{lr} - P^i_{kh|l|s|n}\dot\partial_j\Gamma^{*t}_{mr} - P^i_{kh|l|m|s}\dot\partial_j\Gamma^{*t}_{nr} - \dot\partial_s(P^i_{kh|\ell|m|n})P^t_{jr} = \dot\partial_j\alpha_{\ell mnr}\,P^i_{kh} + \alpha_{\ell mnr}\,P^i_{jkh} + \dot\partial_j\beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \;.$$

Also, applying the equ. by (1.15) for $\left(P^i_{kh|\ell|m}\right)$ in (2.9), we get

(2.10) $$\left\{\dot\partial_j\left(P^i_{kh|\ell|m}\right)\right\}_{|n|r} + [P^r_{kh|\ell|m}\dot\partial_j\Gamma^{*i}_{sn} - P^i_{sh|\ell|m}\dot\partial_j\Gamma^{*r}_{kn} - P^i_{ks\ell|m}\dot\partial_j\Gamma^{*r}_{hn} - P^i_{kh|s|m}\dot\partial_j\Gamma^{*r}_{ln} - \dot\partial_s P^i_{kh|\ell|m}P^r_{jn}\,]_{|r} + P^r_{kh|\ell|m|n}\dot\partial_j\Gamma^{*t}_{sr} - P^i_{sh|\ell|m|n}\dot\partial_j\Gamma^{*t}_{kr} - P^i_{ks|\ell|m|n}\dot\partial_j\Gamma^{*t}_{hr} - P^i_{kh|s|m|n}\dot\partial_j\Gamma^{*t}_{\ell r} - P^i_{kh|l|s|n}\dot\partial_j\Gamma^{*t}_{mr} - P^i_{kh|l|m|s}\dot\partial_j\Gamma^{*t}_{nr} - [\left\{\dot\partial_s\left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} + P^q_{kh|\ell|m|n}\dot\partial_s\Gamma^{*i}_{qr} - P^i_{qh|\ell|m|n}\dot\partial_s\Gamma^{*q}_{kr} - P^i_{kq|\ell|m|n}\dot\partial_s\Gamma^{*q}_{hr} - P^i_{kh|q|m|n}\dot\partial_s\Gamma^{*q}_{lr} - P^i_{kh|l|q|n}\dot\partial_s\Gamma^{*q}_{mr} - P^i_{kh|l|m|q}\dot\partial_s\Gamma^{*q}_{nr} - \dot\partial_q P^i_{kh|\ell|m}P^q_{sn}\,P^t_{jr} = \dot\partial_j\alpha_{\ell mnr}P^i_{kh} + \alpha_{\ell mnr}P^i_{jkh} + \dot\partial_j\beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \;.$$

Again, applying the equ. by (1.15) for $\left(P^i_{kh|\ell}\right)$ in (2.10), we get

(2.11) $$(\dot\partial_j P^i_{kh|l})_{|m|n|r} + \{P^r_{kh|l}\dot\partial_j\Gamma^{*i}_{sm} - P^i_{sh|l}\dot\partial_j\Gamma^{*t}_{km} - P^i_{ks|l}\dot\partial_j\Gamma^{*t}_{hm} - P^i_{kh|s}\dot\partial_j\Gamma^{*t}_{lm} - \dot\partial_s P^i_{kh|l}P^t_{jm}\,\}_{|n|r} + [P^t_{kh|\ell|m}\dot\partial_j\Gamma^{*i}_{rn} - P^i_{sh|\ell|m}\dot\partial_j\Gamma^{*t}_{kn} - P^i_{k|s\ell|m}\dot\partial_j\Gamma^{*t}_{hn} - P^i_{kh|s|m}\dot\partial_j\Gamma^{*t}_{ln} - \dot\partial_s P^i_{kh|\ell|m}P^t_{jn}\,]_{|r} + P^t_{kh|\ell|m|n}\dot\partial_j\Gamma^{*i}_{sr} - P^i_{sh|\ell|m|n}\dot\partial_j\Gamma^{*t}_{kr} - P^i_{ks|\ell|m|n}\dot\partial_j\Gamma^{*t}_{hr} - P^i_{kh|s|m|n}\dot\partial_j\Gamma^{*t}_{\ell r} - P^i_{kh|l|s|n}\dot\partial_j\Gamma^{*t}_{ms} - P^i_{kh|l|m|s}\dot\partial_j\Gamma^{*t}_{nr} - [\left\{\dot\partial_s\left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} + P^q_{kh|\ell|m|n}\dot\partial_s\Gamma^{*i}_{qr} - P^i_{qh|\ell|m|n}\dot\partial_s\Gamma^{*q}_{kr} - P^i_{kq|\ell|m|n}\dot\partial_s\Gamma^{*q}_{hr} - P^i_{kh|q|m|n}\dot\partial_s\Gamma^{*q}_{lr} - P^i_{kh|l|q|n}\dot\partial_s\Gamma^{*q}_{mr} - P^i_{kh|l|m|q}\dot\partial_s\Gamma^{*q}_{nr} - \dot\partial_q P^i_{kh|\ell|m}P^q_{sn}\,P^t_{jr} = \dot\partial_j\alpha_{\ell mnr}P^i_{kh} + \alpha_{\ell mnr}P^i_{jkh} + \dot\partial_j\beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \;.$$

Further, applying the equ. by (1.15) for (P^i_{kh}) in (2.11), we get

(2.12) $$(\dot\partial_j P^i_{kh})_{|l|m|n|r} + \{P^t_{kh}(\dot\partial_j\Gamma^{*i}_{sl}) - P^i_{sh}(\dot\partial_j\Gamma^{*t}_{kl}) - P^i_{ks}(\dot\partial_j\Gamma^{*t}_{hl}) - (\dot\partial_s P^i_{kh})\,P^t_{jl}\}_{|m|n|r} + \{P^t_{kh|l}(\dot\partial_j\Gamma^{*i}_{sm}) - P^i_{sh|l}(\dot\partial_j\Gamma^{*t}_{km}) - P^i_{ks|l}(\dot\partial_j\Gamma^{*t}_{hm}) - P^i_{kh|s}(\dot\partial_j\Gamma^{*t}_{lm}) - (\dot\partial_s P^i_{kh|l})P^t_{jm}\}_{|n|r} + [P^t_{kh|\ell|m}(\dot\partial_j\Gamma^{*i}_{sn}) - P^i_{sh|\ell|m}(\dot\partial_j\Gamma^{*t}_{kn}) - P^i_{ks|\ell|m}(\dot\partial_j\Gamma^{*t}_{hn}) - P^i_{kh|s|m}(\dot\partial_j\Gamma^{*t}_{ln}) - \dot\partial_s\left(P^i_{kh|\ell|m}\right)P^t_{jn}\,]_{|r} + P^r_{kh|\ell|m|n}(\dot\partial_j\Gamma^{*i}_{sr}) - P^i_{sh|\ell|m|n}(\dot\partial_j\Gamma^{*t}_{kr}) - P^i_{ks|\ell|m|n}(\dot\partial_j\Gamma^{*t}_{hr}) - P^i_{kh|s|m|n}(\dot\partial_j\Gamma^{*t}_{\ell r}) - P^i_{kh|l|s|n}(\dot\partial_j\Gamma^{*t}_{mr}) - P^i_{kh|l|m|s}(\dot\partial_j\Gamma^{*t}_{nr}) - [\left\{\dot\partial_s\left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} + P^q_{kh|\ell|m|n}(\dot\partial_s\Gamma^{*i}_{qr}) - P^i_{qh|\ell|m|n}(\dot\partial_s\Gamma^{*q}_{kr}) - P^i_{kq|\ell|m|n}(\dot\partial_s\Gamma^{*q}_{hr}) - P^i_{kh|q|m|n}(\dot\partial_s\Gamma^{*q}_{lr}) - P^i_{kh|l|q|n}(\dot\partial_s\Gamma^{*q}_{mr}) - P^i_{kh|l|m|q}(\dot\partial_s\Gamma^{*q}_{nr}) - \dot\partial_q(P^i_{kh|\ell|m})P^q_{sn}]\,P^t_{jr} = (\dot\partial_j\alpha_{\ell mnr})\,P^i_{kh}$$

$$+\alpha_{\ell mnr} P^i_{jkh} + (\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \quad .$$

Using (1.17b) in (2.12), we get

(2.13)
$$P^i_{jkh|l|m|n|r} + \{P^s_{kh}\dot{\partial}_j\Gamma^{*i}_{sl} - P^i_{sh}\dot{\partial}_j\Gamma^{*s}_{kl} - P^i_{ks}\dot{\partial}_j\Gamma^{*s}_{hl} - \dot{\partial}_s P^i_{kh} P^s_{jl}\}_{|m|n|r}$$
$$+\{P^s_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{sm}) - P^i_{sh|l}(\dot{\partial}_j\Gamma^{*s}_{km}) - P^i_{ks|l}(\dot{\partial}_j\Gamma^{*s}_{hm}) - P^i_{kh|s}\dot{\partial}_j\Gamma^{*s}_{lm} - (\dot{\partial}_s P^i_{kh|l})P^s_{jm}\}_{|n|r}$$
$$+[P^s_{kh|\ell|m}\dot{\partial}_j\Gamma^{*i}_{sn} - P^i_{sh|\ell|m}\dot{\partial}_j\Gamma^{*s}_{kn} - P^i_{ks|\ell|m}\dot{\partial}_j\Gamma^{*s}_{hn} - P^i_{kh|s|m}\dot{\partial}_j\Gamma^{*s}_{ln}$$
$$-\dot{\partial}_s P^i_{kh|\ell|m} P^s_{jn}]_{|r} + P^s_{kh|\ell|m|n}\dot{\partial}_j\Gamma^{*i}_{sr} - P^i_{sh|\ell|m|n}\dot{\partial}_j\Gamma^{*s}_{kr} - P^i_{kr|\ell|m|n}\dot{\partial}_j\Gamma^{*s}_{hr}$$
$$-P^i_{kh|s|m|n}\dot{\partial}_j\Gamma^{*s}_{\ell r} - P^i_{kh|l|s|n}\dot{\partial}_j\Gamma^{*s}_{mr} - P^i_{kh|l|m|s}\dot{\partial}_j\Gamma^{*s}_{nr} - [\{\dot{\partial}_s P^i_{kh|\ell|m|n}\}_{|r}$$
$$+P^q_{kh|\ell|m|n}\dot{\partial}_s\Gamma^{*i}_{qr} - P^i_{qh|\ell|m|n}\dot{\partial}_s\Gamma^{*q}_{kr} - P^i_{kq|\ell|m|n}\dot{\partial}_s\Gamma^{*q}_{hr} - P^i_{kh|q|m|n}\dot{\partial}_s\Gamma^{*q}_{lr}$$
$$-P^i_{kh|l|q|n}\dot{\partial}_s\Gamma^{*q}_{mr} - P^i_{kh|l|m|q}\dot{\partial}_s\Gamma^{*q}_{nr} - \dot{\partial}_q P^i_{kh|\ell|m} P^q_{sn} P^s_{jr} = \dot{\partial}_j\alpha_{\ell mnr} P^i_{kh}$$
$$+\alpha_{\ell mnr} P^i_{jkh} + \dot{\partial}_j\beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \quad .$$

This shows that

$$P^i_{jkh|\ell|m|n|r} = \alpha_{\ell mnr} P^i_{jkh} + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \ ,$$

if and only if

(2.14)
$$\{P^s_{kh}\dot{\partial}_j\Gamma^{*i}_{sl} - P^i_{sh}\dot{\partial}_j\Gamma^{*s}_{kl} - P^i_{ks}\dot{\partial}_j\Gamma^{*s}_{hl} - \dot{\partial}_s P^i_{kh} P^s_{jl}\}_{|m|n|r}$$
$$+\{P^s_{kh|l}\dot{\partial}_j\Gamma^{*i}_{sm} - P^i_{sh|l}\dot{\partial}_j\Gamma^{*s}_{km} - P^i_{ks|l}\dot{\partial}_j\Gamma^{*s}_{hm} - P^i_{kh|s}\dot{\partial}_j\Gamma^{*s}_{lm} - (\dot{\partial}_s P^i_{kh|l})P^s_{jm}\}_{|n|r}$$
$$+[P^s_{kh|\ell|m}\dot{\partial}_j\Gamma^{*i}_{sn} - P^i_{sh|\ell|m}\dot{\partial}_j\Gamma^{*s}_{kn} - P^i_{ks|\ell|m}\dot{\partial}_j\Gamma^{*s}_{hn} - P^i_{kh|s|m}\dot{\partial}_j\Gamma^{*s}_{ln}$$
$$-\dot{\partial}_s(P^i_{kh|\ell|m}) P^s_{jn}]_{|r} + P^s_{kh|\ell|m|n}\dot{\partial}_j\Gamma^{*i}_{sr} - P^i_{sh|\ell|m|n}\dot{\partial}_j\Gamma^{*s}_{kr} - P^i_{kr|\ell|m|n}\dot{\partial}_j\Gamma^{*s}_{hr}$$
$$-P^i_{kh|s|m|n}\dot{\partial}_j\Gamma^{*s}_{\ell r} - P^i_{kh|l|s|n}\dot{\partial}_j\Gamma^{*s}_{mr} - P^i_{kh|l|m|s}\dot{\partial}_j\Gamma^{*s}_{nr} - [\{\dot{\partial}_s P^i_{kh|\ell|m|n}\}_{|r}$$
$$+P^q_{kh|\ell|m|n}\dot{\partial}_s\Gamma^{*i}_{qr} - P^i_{qh|\ell|m|n}\dot{\partial}_s\Gamma^{*q}_{kr} - P^i_{kq|\ell|m|n}\dot{\partial}_s\Gamma^{*q}_{hr} - P^i_{kh|q|m|n}\dot{\partial}_s\Gamma^{*q}_{lr}$$
$$-P^i_{kh|l|q|n}\dot{\partial}_s\Gamma^{*q}_{mr} - P^i_{kh|l|m|q}\dot{\partial}_s\Gamma^{*q}_{nr} - \dot{\partial}_q(P^i_{kh|\ell|m}) P^q_{sn} P^s_{jr} = (\dot{\partial}_j\alpha_{\ell mnr}) P^i_{kh}$$
$$+\dot{\partial}_j\beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) \quad .$$

Hence, we have the following theorem

Theorem 2.3. In P^h-G-FRF$_n$, at the differentiation of the tensor P^i_{jkh} is a generalized four-recurrent if and only if the condition (2.14) holds good.

Interpretation: Theorem 2.3 establishes that fourth-order recurrence is completely characterized by condition (2.14), which acts as an integrability constraint on the curvature tensor.

Transvecting (2.12) by g_{ip} , using (1.3a), (1.9a), (1.19b) and (1.17b), we get

(2.15)
$$P_{jpkh|\ell|m|n|r} + \{g_{ip} P^s_{kh}(\dot{\partial}_j\Gamma^{*i}_{sl}) - P_{sph}(\dot{\partial}_j\Gamma^{*s}_{kl}) - P_{kps}(\dot{\partial}_j\Gamma^{*s}_{hl}) - g_{ip}(\dot{\partial}_s P^i_{kh})P^s_{jl}\}_{|m|n|r}$$
$$+\{g_{ip} P^s_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{sm}) - P_{sph|l}(\dot{\partial}_j\Gamma^{*s}_{km}) - P_{kps|l}(\dot{\partial}_j\Gamma^{*s}_{hm}) - P_{kph|s}(\dot{\partial}_j\Gamma^{*s}_{lm})$$
$$-g_{ip}(\dot{\partial}_s P^i_{kh|l})P^s_{jm}\}_{|n|r} + [g_{ip} P^s_{kh|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{sn}) - P_{sph|\ell|m}(\dot{\partial}_j\Gamma^{*s}_{kn}) - P_{kps|\ell|m}(\dot{\partial}_j\Gamma^{*s}_{hn})$$
$$-P_{kph|s|m}(\dot{\partial}_j\Gamma^{*s}_{ln}) - P_{kph|m|s}(\dot{\partial}_j\Gamma^{*s}_{mn}) - g_{ip}\dot{\partial}_s(P^i_{kh|\ell|m}) P^s_{jn}]_{|r} + g_{ip} P^s_{kh|\ell|m|n}(\dot{\partial}_j\Gamma^{*i}_{sr})$$
$$-P_{sph|m|n}(\dot{\partial}_j\Gamma^{*s}_{kr}) - P_{kps|l|m|n}(\dot{\partial}_j\Gamma^{*s}_{hr}) - P_{kph|s|m|n}(\dot{\partial}_j\Gamma^{*s}_{\ell r}) - P_{kph|l|s|n}(\dot{\partial}_j\Gamma^{*s}_{mr})$$
$$-P_{kph|l|m|s}(\dot{\partial}_j\Gamma^{*s}_{nr}) - \{g_{ip}\dot{\partial}_s(P^i_{kh|\ell|m|n})\}_{|r} + g_{ip}P^q_{kh|\ell|m|n}(\dot{\partial}_s\Gamma^{*i}_{qr}) - P_{qph|l|m|n}(\dot{\partial}_s\Gamma^{*q}_{kr})$$
$$-P_{kpq|l|m|n}(\dot{\partial}_s\Gamma^{*q}_{hr}) - P_{kph|q|m|n}(\dot{\partial}_s\Gamma^{*q}_{lr}) - P_{kph|l|q|n}(\dot{\partial}_s\Gamma^{*q}_{mr}) - P_{kph|l|m|q}(\dot{\partial}_s\Gamma^{*q}_{nr})$$
$$-g_{ip}\dot{\partial}_q(P^i_{kh|\ell|m})P^q_{sn} P^t_{jr} = (\dot{\partial}_j\alpha_{\ell mnr}) P_{kph} + \alpha_{\ell mnr} P_{jpkh}$$
$$+g_{ip}(\dot{\partial}_j\beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(g_{kp} g_{jh} - g_{hp} g_{jk}) \quad .$$

This shows that

(2.16)
$$P_{jpkh|\ell|m|n|r} = \alpha_{\ell mnr} P_{jpkh} + \beta_{\ell mnr}(g_{kp} g_{jh} - g_{hp} g_{jk})$$

if and only if

(2.17)
$$\{g_{ip} P^s_{kh}(\dot{\partial}_j\Gamma^{*i}_{sl}) - P_{sph}(\dot{\partial}_j\Gamma^{*s}_{kl}) - P_{kps}(\dot{\partial}_j\Gamma^{*s}_{hl}) - g_{ip}(\dot{\partial}_s P^i_{kh})P^s_{jl}\}_{|m|n|r}$$
$$+\{g_{ip} P^s_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{sm}) - P_{sph|l}(\dot{\partial}_j\Gamma^{*s}_{km}) - P_{kps|l}(\dot{\partial}_j\Gamma^{*s}_{hm}) - P_{kph|s}(\dot{\partial}_j\Gamma^{*s}_{lm})$$
$$-g_{ip}(\dot{\partial}_s P^i_{kh|l})P^s_{jm}\}_{|n|r} + [g_{ip} P^s_{kh|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{sn}) - P_{sph|\ell|m}(\dot{\partial}_j\Gamma^{*s}_{kn}) - P_{kps|\ell|m}(\dot{\partial}_j\Gamma^{*s}_{hn})$$
$$-P_{kph|s|m}(\dot{\partial}_j\Gamma^{*s}_{ln}) - P_{kph|m|s}(\dot{\partial}_j\Gamma^{*s}_{mn}) - g_{ip}\dot{\partial}_s(P^i_{kh|\ell|m}) P^s_{jn}]_{|r} + g_{ip} P^s_{kh|\ell|m|n}(\dot{\partial}_j\Gamma^{*i}_{sr})$$
$$-P_{sph|m|n}(\dot{\partial}_j\Gamma^{*s}_{kr}) - P_{kps|l|m|n}(\dot{\partial}_j\Gamma^{*s}_{hr}) - P_{kph|s|m|n}(\dot{\partial}_j\Gamma^{*s}_{\ell r}) - P_{kph|l|s|n}(\dot{\partial}_j\Gamma^{*s}_{mr})$$
$$-P_{kph|l|m|s}(\dot{\partial}_j\Gamma^{*s}_{nr}) - \{g_{ip}\dot{\partial}_s(P^i_{kh|\ell|m|n})\}_{|r} + g_{ip}P^q_{kh|\ell|m|n}(\dot{\partial}_s\Gamma^{*i}_{qr}) - P_{qph|l|m|n}(\dot{\partial}_s\Gamma^{*q}_{kr})$$
$$-P_{kpq|l|m|n}(\dot{\partial}_s\Gamma^{*q}_{hr}) - P_{kph|q|m|n}(\dot{\partial}_s\Gamma^{*q}_{lr}) - P_{kph|l|q|n}(\dot{\partial}_s\Gamma^{*q}_{mr}) - P_{kph|l|m|q}(\dot{\partial}_s\Gamma^{*q}_{nr})$$
$$-g_{ip}\dot{\partial}_q(P^i_{kh|\ell|m})P^q_{sn} P^t_{jr} = (\dot{\partial}_j\alpha_{\ell mnr}) P_{kph} + g_{ip}(\dot{\partial}_j\beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) \quad .$$

Consequently, the following theorem is proven.

Theorem 2.4. If and only if the condition (2.16) is met, the differentiated associative tensor is P_{jpkh} of tensor P^i_{jkh} in P^h-G-FRF$_n$ is a generalized four-recurrent.

3. Regarding Generalized P^h-Four-Recurrent Affinely Connected Space

Differential geometry has relied heavily on the study of recurrent spaces. With an emphasis on four-recurrent-affinely connected spaces, we present a novel generalization in this study. Our study intends to further the existing investigation of higher-order recurring structures and their possible uses in engineering and physics. This section will present a new definition for P^h-G-FRF_n, which also has the characteristics of an affinely connected space.

Definition 3.1. An affinely connected space is defined as Finsler space F_n , whose coefficient of parameter, G^m_{nk} , is independent on y^m, and the related equations.

(3.1) a) $G^i_{jmn}=0$ and b) $C_{ijm|n}=0$.

The directional argument has no bearing on the coefficient's parameters Γ^{*i}_{kh} and G^i_{kh} .

(3.2) a) $\dot{\partial}_j G^i_{mn}=0$ and b) $\dot{\partial}_j \Gamma^{*i}_{mn}=o$.

Definition 3.2: A generalized P^h -four-recurrent affinely connected Finsler space is an affinely connected Finsler space satisfying the fourth-order recurrence condition (2.4), together with the directional independence conditions (3.1) - (3.2).

Remark 3.1: Cartan's second curvature tensor P^i_{jkh} , which has the property of P^h-G-FRF_n-affinely connected space, can be referred to as a generalized h-four-recurrent tensor.

Let's examine the affinely connected space P^h-G-FRF_n.

The proof follows by substituting the affine connection conditions (3.1) into the general recurrence identities derived in Section 2.

Considering definition 3.2 and theorem 2.1, we can conclude

Theorem 3.1. The generalized P^h -birecurrent affinely connected space in a ffinely connected generalized P^h -recurrent space is P^h-G-FRF_n-affinely connected.

Applying (3.2b) to (2.12), we obtain

(3.3) $$P^i_{jkh|l|m|n|r} - \{(\dot{\partial}_s P^i_{kh})\, P^s_{jl}\}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_{jm}\}_{|n|r} - [\dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_{jn}]_{|r} - \left[\left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_{jr} = (\dot{\partial}_j \alpha_{\ell mnr}) P^i_{kh} + \alpha_{\ell mnr} P^i_{jkh} + (\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) .$$

This shows that

(3.4) $$P^i_{jkh|l|m|n|r} = \alpha_{\ell mnr} P^i_{jkh} + \beta_{\ell mnr}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) ,$$

if and only if

(3.5) $$\{(\dot{\partial}_s P^i_{kh})\, P^s_{jl}\}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_{jm}\}_{|n|r} - [\dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_{jn}]_{|r} - \left[\left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_{jr} = (\dot{\partial}_j \alpha_{\ell mnr}) P^i_{kh} + \alpha_{\ell mnr} P^i_{jkh} + (\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k) = 0 .$$

Further, using (3.2b) in (2.12), we get

(3.6) $$P_{jpkh|\ell|m|n|r} - \{ g_{ip}\, (\dot{\partial}_s P^i_{kh})\, P^s_{jl} \}_{|m|n|r} - \{ g_{ip}\, (\dot{\partial}_s P^i_{kh|l})\, P^s_{jm}\}_{|n|r} - [\, g_{ip}\, \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_{jn}]_{|r} - \left[\left\{ g_{ip} \dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - g_{ip}\, \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_{jr} = (\dot{\partial}_j \alpha_{\ell mnr})\, P_{kph} + \alpha_{\ell mnr} P_{jpkh} + g_{ip}\, (\dot{\partial}_j \beta_{\ell mnr})(\, \delta^i_k y_h - \delta^i_h y_k) + \beta_{\ell mnr}(g_{kp}\, g_{jh} - g_{hp} g_{jk}) .$$

This shows that

(3.7) $$P_{jpkh|\ell|m|n|r} = \alpha_{\ell mnr}\, P_{jpkh} + \beta_{\ell mnr}(\, g_{kp}\, g_{jh} - g_{hp}\, g_{jk}\,)$$

if and only if

(3.8) $$\{ g_{ip}\, (\dot{\partial}_s P^i_{kh})\, P^s_{jl} \}_{|m|n|r} - \{ g_{ip}\, (\dot{\partial}_s P^i_{kh|l})\, P^s_{jm}\}_{|n|r} - [\, g_{ip}\, \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_{jn}]_{|r} - \left[\left\{ g_{ip} \dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - g_{ip}\, \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_{jr} + (\dot{\partial}_j \alpha_{\ell mnr})\, P_{kph} + \alpha_{\ell mnr} P_{jpkh} + g_{ip}\, (\dot{\partial}_j \beta_{\ell mnr})(\, \delta^i_k y_h - \delta^i_h y_k) .$$

Consequently, the following theorem can be stated.

Theorem 3.2. P^i_{kjh} and its associative P_{jpkh} curvature tensor is generalized trirecurrent tensors in P^h-G-FRF_n-affinely connected space if and only if the requirements (3.4) and (3.7), respectively, are met.

Using (1.5a), (1.7a), (1,9b), (1.19a), and (1.20) to transvect (3.3) by y^j, we obtain

(3.9) $$P^i_{kh|l|m|n|r} - \{(\dot{\partial}_s P^i_{kh}) P^s_l\}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - [\, \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_n]_{|r} - \left[\left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P^i_{kh}\, y^j + \alpha_{\ell mnr} P^i_{kh} + (\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k)\, y^j + \beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) .$$

This shows that

(3.10) $$P^i_{kh|\ell|m|n|r} = \alpha_{\ell mnr} P^i_{kh} + \beta_{\ell mnr}(\delta^i_k y_h - \delta^i_h y_k) ,$$

if and only if

(3.11) $$\{(\dot{\partial}_s P^i_{kh}) P^s_l\}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - [\, \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_n]_{|r} - \left[\left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P^i_{kh}\, y^j + (\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k)\, y^j .$$

Transvecting (3.9) by g_{it} , using (1.3a), (1.9a) and (1.19b), we get

(3.12) $$P_{kth|l|m|n|r} - g_{it}\{(\dot{\partial}_s P^i_{kh}) P^s_l\}_{|m|n|r} - g_{it}\{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - [g_{it} \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_n]_{|r} - \left[g_{it} \left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P_{kth}\, y^j + \alpha_{\ell mnr}\, P_{kth} + g_{it}(\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k)\, y^j + \beta_{\ell mnr}(\, g_{kt}\, y_h - g_{ht}\, y_k) .$$

This shows that

(3.13) $$P_{kth|\ell|m|n|r} = \alpha_{\ell mnr} P_{kth} + \beta_{\ell mnr}(\, g_{kt} y_h - g_{ht} y_k\,)$$

if and only if

(3.14) $$g_{it}\{(\dot{\partial}_s P^i_{kh}) P^s_l\}_{|m|n|r} - g_{it}\{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - [g_{it} \dot{\partial}_s \left(P^i_{kh|\ell|m}\right) P^s_n]_{|r} - \left[g_{it} \left\{\dot{\partial}_s \left(P^i_{kh|\ell|m|n}\right)\right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}\right] P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P_{kth}\, y^j + g_{it}(\dot{\partial}_j \beta_{\ell mnr})(\delta^i_k y_h - \delta^i_h y_k)\, y^j .$$

Transvecting (3.9) by y^k , using (1.5a), (1.6a), (1.7a) and (1.19a), we get

https://journals.ust.edu/index.php/JST https://doi.org/10.20428/jst.v31i2.3635

(3.15) $P^i_{h|l|m|n|s} - \{ (\dot{\partial}_s P^i_{kh}) P^s_l y^k \}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - y^k [\, \dot{\partial}_s \left(P^i_{kh|\ell|m} \right) P^s_n \,]_{|r}$

$$- y^k [\left\{ \dot{\partial}_s \left(P^i_{kh|\ell|m|n} \right) \right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}]\, P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P^i_h\, y^j + \alpha_{\ell mnr}\, P^i_h + (\dot{\partial}_j \beta_{\ell mnr})(y_h\, y^k - \delta^i_h\, F^2)\, y^j + \beta_{\ell mnr}(y_h\, y^k - \delta^i_h\, F^2) \; .$$

This shows that

(3.16) $$P^i_{h|l|m|n|r} = \alpha_{\ell mnr} P^i_h + \beta_{\ell mnr}(y_h\, y^k - \delta^i_h\, F^2) \; ,$$

if and only if

(3.17) $\{(\dot{\partial}_s P^i_{kh}) P^s_l y^k\}_{|m|n|r} - \{(\dot{\partial}_s P^i_{kh|l}) P^s_m\}_{|n|r} - y^k [\, \dot{\partial}_s \left(P^i_{kh|\ell|m} \right) P^s_n \,]_{|r}$

$$- y^k [\left\{ \dot{\partial}_s \left(P^i_{kh|\ell|m|n} \right) \right\}_{|r} - \dot{\partial}_q (P^i_{kh|\ell|m}) P^q_{sn}]\, P^s_r = (\dot{\partial}_j \alpha_{\ell mnr})\, P^i_h\, y^j + (\dot{\partial}_j \beta_{\ell mnr})(y_h\, y^k - \delta^i_h\, F^2)\, y^j \; .$$

Thus, the following theorem can be deduced.

Theorem 3.3. The h-covariant derivative of fourth order for the torsion tensor P^i_{kh} , its associative tensor P_{krh} , and the deviation tensor P^i_h in P^h-G-FRF$_n$-affinely connected space is given by (3.10), (3.13), and (3.16) if and only if the conditions (3.11), (3.14), and (3.17), respectively, hold.

4. Motivation and Potential Applications

This research is driven by the growing complexity of modern financial systems operating in unstable economies and rapidly evolving digital landscapes. The digital transformation brought forth by fintech has made banking sectors in emerging and post-crisis economies more vulnerable to systemic interdependencies, asymmetric risk transmission, and nonlinear financial dynamics.

By fusing complex Finsler geometry with current challenges in FinTech-enabled financial systems, this paper provides a basic mathematical viewpoint that guides future application research and theoretical development. For banking companies that operate throughout times of economic recovery and the digital revolution, where stability, resilience, and effective risk management are essential to long-term financial survival, the recommended approach is particularly appropriate.

Conclusion

This paper develops an advanced Finsler geometric framework to assess banking stability amidst FinTech innovation and digital transformation. By analyzing fourth-order curvature tensors on P^h -generalized four-recurrent Finsler manifolds, it establishes a rigorous mathematical foundation for modeling higher-order interactions, asymmetric risk transmission, and nonlinear dynamics in modern banking, particularly within emerging and post-crisis economies.

The direction-dependent nature of Finsler geometry effectively captures partial stability patterns, cyclical irregularities, and systemic risk propagation. Furthermore, generalized recurrence conditions provide insights into structural resilience during economic recovery. While primarily theoretical, this framework enhances risk modeling and dynamic stress testing, fostering interdisciplinary integration between differential geometry and applied financial analysis to ensure long-term financial survival.

The proposed framework opens the door for developing new stress-testing algorithms based on non-Riemannian geometry.

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