Computational Extensions of Generalized R^h-Recurrent Finsler Spaces and Applications to Geometric Machine Learning

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Received: ١٢/١٢/2025
Revised: ١٦/١٢/2025
Accepted: 30/12/2025

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Abstract**—** We investigate generalized R^h-recurrent Finsler spaces defined via a recurrence condition on Cartan's third curvature tensor. We derive identities involving Cartan torsion, Berwald torsion H^i_{kh} , the deviation tensor H^i_h , and Ricci-type contractions. In particular, we prove non-degeneracy results showing that the Ricci tensor, curvature vector, deviation tensor, and scalar curvature are necessarily non-vanishing. We also provide equivalent characterizations of generalized R^h-recurrence through contracted forms of the defining condition. Finally, we outline how these identities can support curvature-aware numerical schemes for manifold optimization and geometric machine learning in anisotropic settings.

Keywords— *Finsler geometry, generalized recurrence, Cartan connection, curvature identities, manifold optimization, geometric machine learning.*

1. Introduction

Finsler geometry offers a potent mathematical tool for simulating anisotropic structures by adding direction-dependent metrics to the traditional Riemannian framework by Rund. In the last ten years, Finsler geometry has become more popular in computational domains like geometric machine learning, optimization on manifolds, and scientific simulation, in addition to pure mathematics. The simplified curvature relations of recurrent manifold structures make curvature calculation easier to handle in numerical techniques. Nevertheless, generalized R^h-recurrent Finsler structures have not received much attention because the majority of research has been on conventional recurrent circumstances.

In this study, a thorough analysis of generalized R^h-recurrent Finsler spaces is presented, along with novel tensorial identities, non-vanishing conditions for curvature variables, and a computational interpretation that links these geometric structures to machine learning. As a result, there is a natural link between developing computational applications and abstract differential geometry.

Related Work

In Riemannian and Finsler settings, recurrent structures in differential geometry have been thoroughly examined. Curvature identities, torsion relations, and the characterization of particular Finsler spaces like Landsberg, Berwald, and C^h-recurrent spaces were the main topics of Cartan, Matsumoto, and later academics' classical studies.

Studies on Ricci-type identities in Finsler manifolds, generalized recurrence conditions, and higher-order recurrent tensors are examples of recent contributions. Nevertheless, very little research has looked at generalized R^h-recurrent spaces. The complete set of identities described here (Sections 2 and 3) has not been derived by anyone. Generalized recurrent Finsler structures have not before been linked to computational frameworks, especially geometric machine learning.

This demonstrates the current work's uniqueness and transdisciplinary value.

Preliminaries

This section provides an overview of the key ideas that are used in the article, such as:

- Finsler metric $F(x, y)$.
- Cartan connection.
- h- and v-covariant derivatives.
- Cartan torsion C_{ijk} .
- $h(v)$-torsion tensor H^i_{kh} .
- Curvature tensor R^i_{jkh} .
- Ricci and scalar curvature components.

Notation. Let $F(x, y)$be a Finsler metric on an n-dimensional manifold M. We use $\partial_k = \frac{\partial}{\partial x^k}$ and $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The symbol "$|\,k$" denotes the h-covariant derivative with respect to Cartan's connection. Indices are raised and lowered using g_{ij} and its inverse g^{ij}, and summation over repeated indices is assumed.

The fundamental foundation for this study has been established by earlier research on recurring and generalized curvature structures in Finsler geometry. The fundamental ideas of generalized recurrence and curvature decomposition in differential geometry were first introduced by Chaki, Matsumoto, Yano, and Bochner. These concepts were extended to higher-order recurrent Finsler spaces in later publications, such as those by Misra, Pandey, Mishra, Ahsan, Ali, Shaikh, Baishya and others who studied the derivative structures of Berwald and Cartan. Advanced generalizations incorporating W-, R-, and projective curvature tensors were introduced in more recent research by Al-Qashbari and associates, providing deeper understanding of covariant recurrence, tensor decomposition, and higher-order geometric identities.

In addition to improving the categorization of recurrent Finsler manifolds, these initiatives produced strong algebraic tools for managing intricate curvature connections. The current work expands the theory of generalized R^h -recurrence and links its structural characteristics to contemporary computing frameworks, building on this literature and enhancing geometric theory and its new applications.

Let us examine an n-dimensional Finsler space that satisfies the necessary requirements and has the metric function F.

Let's look at the components of the associated metric tensor g_{ij} , Cartan's connection parameters Γ^{*i}_{jk} , and Berwald's connection parameters G^i_{jk} . These are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments.

The correlation between the corresponding tensor g^{ij} and the two sets of quantities g_{ij} is

(1.1) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases}$$

The following relations are satisfied by the vectors y_i and y^i .

(1.2) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\dot\partial_i\, F^2 = F\dot\partial_i F$ and e) $\dot\partial_j y^i = \delta^i_j$, f) $\delta^i_k g_{ji} = g_{kj}$,

g) $\delta^i_j g^{jk} = g^{ik}$, h) $\delta^i_k y^k = y^k$ and i) $\delta^i_k y_i = y_k$.

The definition of the tensor C_{ijk} is

(1.3) $$C_{ijk} = \frac{1}{2}\dot\partial_i g_{jk} = \frac{1}{4}\dot\partial_i\dot\partial_j\dot\partial_k\, F^2 \;,$$

is referred to as the torsion tensor (h)hv. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all of its indices.

The (v)hv-torsion tensor C^h_{ik} and and its corresponding (h) hv-torsion tensor C_{ijk} have the following relationship:

(1.4) a) $C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0$, b) $C^i_{jk}y^j = C^i_{kj}y^j = 0$

and c) $C^h_{ik} = g^{hj}\, C_{ijk}$.

Additionally, the (v)hv-torsion tensor C^h_{ik} is is positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.

In relation to x^k , É, for any vector field X^i . É. Cartan calculated the h-covariant derivative.

(1.5) $$X^i_{|k} = \partial_k X^i - \left(\dot\partial_r\, X^i\right) G^r_k + X^r \Gamma^{*i}_{rk} \;.$$

The metric tensor g_{ij} and the vector y^i are covariant constants with regard to a process mentioned above.

(1.6) a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{ij}_{|k} = 0$.

The process of h-covariant differentiation with respect to x^k commutes with partial differentiation with respect to y^j for any vector field X^i.

(1.7) $$\dot\partial_j\left(X^i_{|k}\right) - \left(\dot\partial_j X^i\right)_{|k} = X^r\left(\dot\partial_j \Gamma^{*i}_{rk}\right) - \left(\dot\partial_r X^i\right) P^r_{jk}$$

, where

(1.8) a) $\dot\partial_j \Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk}$, b) $P^i_{kh}\, y^k = 0 = P^i_{kh}\, y^h$ and c) $P^i_{jkh}\, y^j = P^i_{kh}$.

The associated tensor P_{jkh} is supplied by the tensor P^i_{kh} , which is referred to as the v(hv)-torsion tensor and the (hv)-curvature tensor P^i_{jkh} has an associate tensor P_{ijkh} that is given by

(1.9) a) $g_{rj}P^r_{kh} = P_{kjh}$ and b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$.

The values H^i_{jkh} and H^i_{kh} form are the components of tensors; they are referred to as the curvature tensor of Berwald and the torsion tensor, respectively, and have the following definitions:

(1.10) a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh}G^i_{rj} + G^i_{rhj}G^r_k - \partial_j G^i_{hk} - G^r_{hk}G^i_{rj} - G^i_{rkj}G^r_h$, and

b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - \partial_k G^i_h - G^r_h\, C^i_{rk}$.

Their lower indices, k and h, are skew-symmetric. Additionally, in their directional arguments, they are degree zero and degree one positively homogenous, respectively.

Additionally, they are connected by

(1.11) a) $H^i_{jkh}y^j = H^i_{kh}$, b) $H^i_{jkh} = \partial_j H^i_{kh}$ and c) $H^i_{jk} = \partial_j\, H^i_k$.

Initially, the mean of the tensor H^i_h , also known as the deviation tensor, was used to generate these tensors.

(1.12) $$H^i_h = 2\partial_h G^i - \partial_r G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h \;.$$

In the directional arguments, the deviation tensor H^i_h is positively homogeneous of degree 2.

The following results from contracting the indices i and h in (1.11) and (1.12) in light of Euler's theorem on homogeneous functions:

(1.13) a) $H^i_{jk}y^j = -H^i_{kj}y^j = H^i_k$ and b) $y_i H^i_j = 0$.

The following is satisfied by the values H^i_{jkh} and H^i_{kh} are

(1.14) a) $H_{ijkh} = g_{jr}\, H^r_{ihk}$, b) $H_{jk.h} = g_{jr}H^r_{hk}$ and c) $y_i\, H^i_{jk} = 0$.

The Bianchi identity is satisfied by Cartan's third curvature tensor R^i_{jkh}.

(1.15) a) $R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + \left(R^r_{mhs}P^i_{jkr} + R^r_{mkh}P^i_{jsr} + R^r_{msk}P^i_{jhr}\right) y^m = 0$

and b) $R_{ijhk} + R_{ihkj} + R_{ikjh} + \left(C_{ijs}\, K^s_{rhk} + C_{ihs}\, K^s_{rkj} + C_{iks}\, K^s_{rjh}\right) y^r = 0$,

where c) $P^r_{ijs} = \dot\partial_s \Gamma^{*r}_{ij} - C^r_{is|j} + C^r_{im}\, C^m_{js|k}\, y^k$.

The curvature scalar R of the curvature tensor R^i_{jkh} , the deviation tensor R^r_h , and the Ricci tensor R_{jk} are all provided by

(1.16) a) $R^i_{jkh} y^j = H^i_{hk} = K^i_{jhk}\, y^j$, b) $R_{ijhk} = g_{rj}\, R^r_{ihk}$,
c) $R_{jkhm}\, y^j = H_{kh.m}$, d) $R^r_{ihk} = g^{jr} R_{ijhk}$
and e) $R^i_{jkh}\, g^{jk} = R^i_h$.

Additionally, the contracted tensors R_{kh} (the Ricci tensor) and R_k (the curvature vector) are linked by

(1.17) a) $R_{jk} y^k = R_j$, b) $R_{jk}\, y^j = H_k$, c) $R^i_{jki} = R_{jk}$ and d) $R^i_i = R$.

The following relation is also satisfied by this tensor.

(1.18) a) $R^i_{jkh} = K^i_{jkh} + C^i_{js}\, K^s_{rhk}\, y^r$, b) $R_{ijkh} = K_{ijkh} + C_{ijs}\, H^s_{kh}$
and c) $R_{jkhm}\, y^j = H_{kh.m}$.

where R_{ijkh} is R^i_{jkh} associated curvature tensor. The Bianchi identities are satisfied by Cartan's fourth curvature tensor, K^i_{jkh}, and its associated curvature tensor, K_{ijkh} .

(1.19) a) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$ and b) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$.

2. Computational Extensions of Generalized R^h -Recurrent Finsler Spaces with Applications to Geometric Machine Learning

Generalized R^h -recurrent Finsler spaces provide an expanded analytical framework for examining Cartan's curvature tensors and their covariant behavior by expanding the classical notion of recurrence in differential geometry. By imposing specific recurrence relations on the third curvature tensor, these structures play a fundamental computational role that makes it possible to characterize geometric quantities more precisely, which is crucial for contemporary applications. This is especially true in geometric machine learning, where curvature-driven invariants and recurrent geometric patterns aid in data representation and feature extraction on manifolds. Because of their reliance on covariant vector fields and metric contractions, generalized R^h-recurrent Finsler spaces provide a methodical handling of horizontal torsion and deviation tensors. Significant geometric implications come from the ensuing structural constraints, including the ensured non-vanishing of important curvature quantities like the scalar curvature, curvature vector, and Ricci tensor. Because they guarantee the resilience of curvature-based learning methods that depend on the persistence of geometric information, these non-degeneracy requirements are particularly useful for computational models.

As a result, researching these recurrence relations improves the theoretical categorization of Finsler spaces and fortifies the computational underpinnings needed to incorporate Finsler geometry into learning systems on curved data manifolds.

Let us examine a Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} meets the following requirements.

(2.1)
$$R^i_{jkh|l} = \lambda_l R^i_{jkh} + \mu_l\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \tfrac{1}{4}\delta_l\left(R^i_h\, g_{jk} - R^i_k\, g_{jh}\right) , \quad R^i_{jkh} \neq 0 ,$$

where λ_l , μ_l and δ_l are fields of non-null covariant vectors. Such a space will be referred to as a generalized R^h-recurrent space. In a nutshell, we will refer to it as GR^h- RF_n.

Using (1.6a), (1.16b), and the metric tensor g_{ip} to transvect (2.1) and in view of (1.2f), we obtain

(2.2)
$$R_{jpkh|l} = \lambda_l R_{jpkh} + \mu_l\left(g_{hp}\, g_{jk} - g_{kp}\, g_{jh}\right) + \tfrac{1}{4}\delta_l\left(R^i_h g_{jk} - R^i_k g_{jh}\right) g_{ip} .$$

On the other hand, the condition (2.1) is obtained by transvecting the condition (2.2) by the associate tensor g^{ip} of the metric tensor g_{ip}. As a result, condition (2.2) is the same as condition (2.1). Consequently, a generalized R^h-recurrent space with condition (2.2).

Thus, we have

Theorem 2.1. The condition (2.2) may be used to describe a GR^h- RF_n .

Applying (1.6b), (1.16a), and (1.2a), to transverse the condition (2.1) by y^j, we obtain

(2.3)
$$H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \tfrac{1}{4}\delta_l\left(R^i_h\, y_k - R^i_k\, y_h\right) .$$

Additionally, using (1.6b), (1.13a), (1.2b), and in light of (1.1), transvecting (2.3) by y^k yields

(2.4)
$$H^i_{h|l} = \lambda_l H^i_h + \mu_l\left(\delta^i_h\, F^2 - y_h\, y^i\right) + \tfrac{1}{4}\delta_l\left(R^i_h\, F^2 - R^i_k\, y_h y^k\right) .$$

Thus, we have

Theorem 2.2. The conditions (2.3) and (2.4) determine the covariant derivative of the deviation tensor H^i_h and the torsion tensor H^i_{kh} in GR^h-RF_n .

Utilizing (1.17c), (1.17d), (1.2f) and (1.1) to contract the indices i and h in condition (2.1), we obtain

(2.5)
$$R_{jk|l} = \lambda_l R_{jk} + (n-1)\, \mu_l\, g_{jk} + \tfrac{1}{4}\delta_l\left(R\, g_{jk} - R^i_k\, g_{ji}\right) .$$

Using (1.6b), (1.17a), and (1.2a) to transvect (2.5) by y^k, we obtain

(2.6)
$$R_{j|l} = \lambda_l\, R_j + (n-1)\, \mu_l\, y_j + \tfrac{1}{4}\delta_l\left(R\, y_j - R^i_k\, g_{ji}\, y^k\right) .$$

Additionally, using (1.6c), (1.16e), (1.2g), and in light of (1.1), we transvect the condition (2.1) by the associate tensor g^{jk} of the metric tensor g_{jk} .

(2.7)
$$R^i_{h|l} = \lambda_l\, R^i_h + (n-1)\mu_l\, \delta^i_h + \tfrac{1}{4}\delta_l\left(nR^i_h - R^i_k\, \delta^k_h\right) .$$

By contracting the indices i and h under conditions (2.7), (1.17d), and (1.1), we obtain

(2.8)
$$R_{|l} = \lambda_l\, R + n\,(n-1)\, \mu_l + \tfrac{1}{4}\delta_l\left(nR - R^i_k\, \delta^k_i\right) .$$

The Ricci tensor R_{jk} , curvature vector R_j , deviation tensor R^i_h , and curvature scalar R of a generalized R^h-recurrent space cannot vanish due to the conditions (2.5), (2.6), (2.7), and (2.8). This is because their vanishing implies the covariant vector field's disappearance μ_l , i.e. $\mu_l = 0$, which is a contradiction.

We therefore draw the conclusion of

Theorem 2.3. The curvature scalar R, the deviation tensor R^i_h , the curvature vector R_j , and the Ricci tensor R_{jk} are all non-vanishing in GR^h-RF_n .

3. Advanced Curvature Identities in Generalized R^h-Recurrent Finsler Spaces: Structural Foundations for Computational Modelling

This section presents the basic geometric identities needed for algorithmic modeling and machine-learning applications, in keeping with the larger goal of creating computational extensions of generalized R h-recurrent Finsler spaces. We offer a strong theoretical framework that facilitates further computational formulations by obtaining higher-order recurrence relations and structural curvature identities. The mathematical foundation for converting geometric recurrence into computable structures consists of identities involving the Cartan tensor, deviation tensor, and different curvature components. These findings not only enhance the theoretical comprehension of GR^h-RF_n . The spaces but also make it possible to create geometric learning algorithms that use representations that are sensitive to curvature. As a result, the analytical advances discussed here provide a crucial link between contemporary computational methods in geometric machine learning and traditional Finsler geometry.

We take covariant differentiation of the formula (1.15b) with regard to x^l in the sense of Cartan using (1.6c), (1.16a), (1.16d), and (1.4c). Then, we transvect (1.15b) by the associate tensor g^{jp} of the metric tensor g_{jp} .

$$R^p_{ihk|l} + g^{jp} R_{ihkj|l} + g^{jp} R_{ikjh|l} + \left(C^p_{is} H^s_{hk} + g^{jp} C_{ihs} H^s_{kj} + g^{jp} C_{iks} H^s_{jh}\right)_{|l} = 0 \quad . \tag{3.1}$$

Using (1.6b), (1.16a), (1.18c), (1.4b), and (1.4a) to transvect (3.1) by y^i , we obtain

$$H^p_{hk|l} + g^{jp} H_{hk.j|l} + g^{jp} H_{kj.h|l} = 0 \quad . \tag{3.2}$$

We therefore draw the conclusion of

Theorem 3.1. The identities (3.1) and (3.2) are valid in GR^h-RF_n .

By applying (1.16b) and (1.16a) to the identity (1.15b), we obtain

$$g_{rj} R^r_{ihk} + g_{rh} R^r_{ikj} + g_{rk} R^r_{ijh} + C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} = 0 \quad . \tag{3.3}$$

Now, using (1.16a) and (1.4b) to transvect (3.3) by y^i, we obtain

$$g_{rj} H^r_{hk} + g_{rh} H^r_{kj} + g_{rk} H^r_{jh} = 0 \quad . \tag{3.4}$$

Equation (3.4), when utilizing (1.14b), produces

$$H_{hj.k} + H_{kh.j} + H_{jk.h} = 0 \quad . \tag{3.5}$$

Using (1.13a) and (1.2a) and (1.14c) to transvect (3.4) by y^h, we obtain

$$g_{rj} H^r_k = g_{rk} H^r_j \quad . \tag{3.6}$$

We therefore draw the conclusion of

Theorem 3.2. The identities (3.4), (3.5), and (3.6) are valid in GR^h-RF_n .

By applying (1.16a) to the identity (1.15a), we obtain

$$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + H^s_{kh} P^r_{ijs} + H^s_{jk} P^r_{ihs} + H^s_{hj} P^r_{iks} = 0 \quad . \tag{3.7}$$

Given the condition (2.1), the identity (3.7) can be expressed as

$$\begin{aligned}&\lambda_h R^r_{ijk} + \lambda_k R^r_{ihj} + \lambda_j R^r_{ikh} + \mu_h\left(\delta^r_k g_{ij} - \delta^r_j g_{ik}\right) + \mu_k\left(\delta^r_j g_{ih} - \delta^r_h g_{ij}\right)\\ &+ \mu_j(\delta^r_h g_{ik} - \delta^r_k g_{ih}) + \tfrac{1}{4}\delta_h\left(R^r_k\, g_{ij} - R^r_j\, g_{ik}\right) + \tfrac{1}{4}\delta_k\left(R^r_j\, g_{ih} - R^r_k\, g_{ij}\right)\\ &+ \tfrac{1}{4}\delta_j(R^r_h\, g_{ik} - R^r_k\, g_{ih}) + \left(H^s_{kh} P^r_{ijs} + H^s_{jk} P^r_{ihs} + H^s_{hj} P^r_{iks}\right) = 0 \quad .\end{aligned} \tag{3.8}$$

Transvecting (3.8) by y^i yields (1.16a), (1.2a), and (1.8c).

$$\begin{aligned}&\lambda_h H^r_{jk} + \lambda_k H^r_{hj} + \lambda_j H^r_{kh} + \mu_h\left(\delta^r_k y_j - \delta^r_j y_k\right) + \mu_k\left(\delta^r_j y_h - \delta^r_h y_j\right)\\ &+ \mu_j(\delta^r_h y_k - \delta^r_k y_h) + \tfrac{1}{4}\delta_h\left(R^r_k\, y_j - R^r_j\, y_k\right) + \tfrac{1}{4}\delta_k\left(R^r_j\, y_h - R^r_k\, y_j\right)\\ &+ \tfrac{1}{4}\delta_j(R^r_h\, y_k - R^r_k\, y_h) + \left(H^s_{kh} P^r_{js} + H^s_{jk} P^r_{hs} + H^s_{hj} P^r_{ks}\right) = 0 \quad .\end{aligned} \tag{3.9}$$

Using y^j to transverse (3.9). (1.13a), (1.2b), (1.1), and (1.8b) are used to get

$$\begin{aligned}&\lambda_h H^r_k - \lambda_k H^r_h + \lambda H^r_{kh} + \mu_h(\delta^r_k F^2 - y_k\, y^r) + \mu_k(y_h\, y^r - \delta^r_h F^2)\\ &+ \mu\, (\delta^r_h\, y_k - \delta^r_k\, y_h) + \tfrac{1}{4}\delta_h\left(R^r_k\, F^2 - R^r_j\, y_k\, y^j\right) + \tfrac{1}{4}\delta_k\left(R^r_j y_h y^j - R^r_k\, F^2\right)\\ &+ \tfrac{1}{4}\delta(R^r_h\, y_k - R^r_k\, y_h) + (H^s_k\, P^r_{hs} - H^s_h\, P^r_{ks}) = 0 \quad ,\end{aligned} \tag{3.10}$$

where $\lambda_j\, y^j = \lambda$, $\mu_j\, y^j = \mu$ and $\delta_j\, y^j = \delta$.

We therefore draw the conclusion of

Theorem 3.3. The identities (3.8), (3.9), and (3.10) are valid in GR^h-RF_n .

Additionally, by transvecting (3.9) and (3.10) by the vector y_r using (1.14c), (1.1), (1.13b), (1.2i), and (1.2b), we obtain

$$\begin{aligned}&\left(H^s_{kh}\, P^r_{js} + H^s_{jk}\, P^r_{hs} + H^s_{hj}\, P^r_{ks} \right) y_r + \tfrac{1}{4}\delta_h\left(R^r_k\, y_j - R^r_j\, y_k\right) y_r\\ &+ \tfrac{1}{4}\delta_k\left(R^r_j\, y_h - R^r_k\, y_j\right) y_r + \tfrac{1}{4}\delta_j(R^r_h\, y_k - R^r_k\, y_h) y_r = 0\end{aligned} \tag{3.11}$$

, and

$$\begin{aligned}&H^s_k\, y_r P^r_{hs} + \tfrac{1}{4}\delta_h\left(R^r_k\, F^2 - R^r_j\, y_k\, y^j\right) y_r + \tfrac{1}{4}\delta_k\left(R^r_j y_h y^j - R^r_k\, F^2\right) y_r\\ &+ \tfrac{1}{4}\delta(R^r_h\, y_k - R^r_k\, y_h)\, y_r - H^s_h\, y_r P^r_{ks} = 0 \quad ,\end{aligned} \tag{3.12}$$

respectively.

We therefore draw the conclusion of

Theorem 3.4. The identity (3.11) is valid in $GR^h\text{-}RF_n$.

Theorem 3.5. We have the identity (3.12) in $GR^h\text{-}RF_n$.

Using (1.16b), (1.1), (1.9b), (1.14b), (1.9a), (1.2f), and (1.2a), we transvect (3.8), (3.9), and (3.10) by the metric tensor g_{rm} ·

$$\lambda_h R_{imjk} + \lambda_k R_{imhj} + \lambda_j R_{imkh} + \mu_h\left(g_{km}\, g_{ij} - g_{jm}\, g_{ik}\right) + \mu_k\left(g_{jm}\, g_{ih} - g_{hm}\, g_{ij}\right) + \mu_j(g_{hm}\, g_{ik} - g_{km}\, g_{ih}) + \frac{1}{4}\delta_h\left(R^r_k\, g_{ij} - R^r_j\, g_{ik}\right) g_{rm} + \frac{1}{4}\delta_k\left(R^r_j\, g_{ih} - R^r_k\, g_{ij}\right) g_{rm} + \frac{1}{4}\delta_j(R^r_h\, g_{ik} - R^r_k\, g_{ih})\, g_{rm} + \left(H^s_{kh} P_{imjs} + H^s_{jk}\, P_{imhs} + H^s_{hj} P_{imks}\right) = 0 \quad , \tag{3.13}$$

$$\lambda_h H_{jm.k} + \lambda_k H_{hm.j} + \lambda_j H_{km.h} + \frac{1}{4}\delta_h\left(R^r_k y_j - R^r_j\, y_k\right) g_{rm} + \frac{1}{4}\delta_k\left(R^r_j\, y_h - R^r_k y_j\right) g_{rm} + \frac{1}{4}\delta_j(R^r_h\, y_k - R^r_k\, y_h)\, g_{rm} + \left(H^s_{kh}\, P_{jms} + H^s_{jk}\, P_{hms} + H^s_{hj}\, P_{kms}\right) = 0 \quad , \tag{3.14}$$

and

$$g_{rm}\,(\lambda_h H^r_k - \lambda_k H^r_h + \lambda\, H^r_{kh}) + \mu_h(g_{km} F^2 - y_k\, y_m) + \mu_k(y_h\, y_m - g_{hm} F^2) + \mu(g_{hm} y_k - g_{km}\, y_h) + \frac{1}{4}\delta_h\left(R^r_k F^2 - R^r_j\, y_k y^j\right) g_{rm} + \frac{1}{4}\delta_k\left(R^r_j\, y_h y^j - R^r_k\, F^2\right) g_{rm} + \frac{1}{4}\delta(R^r_h\, y_k - R^r_k\, y_h)\, g_{rm} + (H^s_k\, P_{hms} - H^s_h\, P_{kms}) = 0 \quad , \tag{3.15}$$

respectively.

We therefore draw the conclusion of

Theorem 3.6. In $GR^h\text{-}RF_n$, the identities (3.13), (3.14), and (3.15) are valid.

In the identity (1.15b), we have (1.18b) and (1.16a)

$$K_{ijhk} + K_{ihkj} + K_{ikjh} + 2\left(C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} \right) = 0 \; . \tag{3.16}$$

In view of (1.19b), the identity (3.16) becomes

$$C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0 \; . \tag{3.17}$$

Using h-covariant differentiation of the identity (3.17) with respect to x^l in Cartan's meaning, we obtain

$$C_{ijs|l} H^s_{hk} + C_{ijs}\, H^s_{hk|l} + C_{ihs|l} H^s_{kj} + C_{ihs}\, H^s_{kj|l} + C_{iks|l}\, H^s_{jh} + C_{iks}\, H^s_{jh|l} = 0 \; . \tag{3.18}$$

Now, using (1.6c) and (1.4c) to transvect (3.18) by the related tensor g^{rj}, we obtain

$$C^r_{is|l}\, H^s_{hk} + C^r_{is}\, H^s_{hk|l} + C_{ihs|l}\, H^s_{kj} g^{rj} + C_{ihs}\, H^s_{kj|l}\, g^{rj} + C_{iks|l}\, H^s_{jh}\, g^{rj} + C_{iks}\, H^s_{jh|l}\, g^{rj} = 0 \; . \tag{3.19}$$

Transvecting (3.18) and (3.19) by y^h , using (1.6b), (1.13a) and (1.4a), we get

$$C_{ijs|l}\, H^s_k + C_{ijs}\, H^s_{k|l} = C_{iks|l}\, H^s_j + C_{iks}\, H^s_{j|l} \quad , \tag{3.20}$$

and

$$C^r_{is|l}\, H^s_k + C^r_{is}\, H^s_{k|l} = \left(C_{iks|l}\, H^s_j + C_{iks}\, H^s_{j|l} \right) g^{rj} \; , \tag{3.21}$$

respectively.

We thus draw the conclusion of

Theorem 3.7. In $GR^h\text{-}RF_n$, the identities (3.18), (3.19), (3.20), and (3.21) are valid.

Additionally, using (1.6a), (1.1) and (1.14b) to transvect (3.18) and (3.19) by g_{rs} , we obtain

$$C_{ijs|l}\, H_{hr.k} + C_{ijs}\, H_{hr.k|l} + C_{ihs|l}\, H_{kr.j} + C_{ihs}\, H_{kr.j|l} + C_{iks|l}\, H_{jr.h} + C_{iks}\, H_{jr.h|l} = 0 \tag{3.22}$$

and

$$C^r_{js|l}\, H_{hr.k} + C^r_{js}\, H_{hr.k|l} + \left(C_{ihs|l}\, H_{kr.j} + C_{ihs}\, H_{kr.j|l} + C_{iks|l}\, H_{jr.h} + C_{iks}\, H_{jr.h|l} \right) g^{rj} = 0 \; , \tag{3.23}$$

respectively.

We thus get the conclusion of

Theorem 3.8. In $GR^h\text{-}RF_n$, the identities (3.22) and (3.23) are valid.

4. Machine Learning Applications and Computational Consequences

4.1. Optimization on Manifolds Considering Curvature

The formulas provided by the recurrence identities allow for:

a) Effective curvature tensor computation in numerical contexts
b) Development of gradient descent techniques based on Finsler;
c) Curvature-informed adaptive learning rates

Utilized in: Riemannian/Finsler optimization; geometric deep learning, and variational models on manifolds.

4.2. Data Geometry Tensorial Structures

Calculating anisotropic measures for high-dimensional data, improving embedding methods (UMAP, t-SNE) using curvature signals, and modeling directional data with Finsler metrics rather than Euclidean/Riemannian ones are all made possible by the obtained identities.

5. Conclusion and Future Work

In this work, extended R^h-recurrent Finsler spaces are introduced and analyzed. Several novel identities and characterizations for curvature and torsion tensors are established. Strong structural constraints that set this class of spaces apart from classical recurrent geometries are provided by the non-vanishing results achieved.

In addition to its theoretical contributions, the work's integration of computational techniques with generalized Finsler recurrence opens up new study avenues. Future research could concentrate on:

a) Developing numerical algorithms based on the derived identities
b) Implementing Finsler-based optimization methods in machine learning

c) Constructing neural architectures that adapt their geometry using curvature recurrence
d) Applying these ideas to computer vision, robotics, and data geometry

Authors' Contributions

Adel Mohammed Ali Al-Qashbari conceived the research idea, performed the analysis and calculations, and wrote the first draft of the manuscript. Qasem M. M. reviewed and edited the final version of the manuscript.
All authors read and approved the final manuscript.

Conflict of Interest

The authors declare that there is no conflict of interest.

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