Generalized Trirecurrence of Cartan's Second Curvature Tensor in Ph-Recurrent Spaces

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Abstract— This paper explores Cartan's second curvature tensor P^i_{jkh} within the framework of generalized recurrent spaces. We derive and analyze the generalized recurrence conditions, and show that P^i_{jkh} satisfies the recurrence relations of the first, second, and third orders. Specifically, we investigate the conditions that lead to the tensor's behavior in generalized P^h-recurrent spaces, denoted as P^h-G-TRF$_n$, and establish several theorems regarding the curvature tensor's covariant derivatives and its relationship with other curvature tensors. Furthermore, we connect these recurrence conditions to the P-Ricci tensor and vector fields in higher-dimensional spaces. We demonstrate the interrelations between these tensors through a series of covariant derivative computations and conclude that the curvature tensor cannot vanish under specific geometric conditions. This study advances the field of differential geometry and its applications in physics by offering a thorough foundation for comprehending the characteristics of Cartan's curvature tensor in generalized recurrent spaces.

Keywords— Tensor P^i_{jkh}-generalized recurrent, Generalized Third Recurrent Space, Affinely Connected Spaces, Cartan's Curvature Tensor, Covariant Derivatives, Recurrence Conditions.

I. INTRODUCTION

Generalized third recurrent affinely connected spaces are a specific class of manifolds that exhibit intriguing properties related to the recurrence and birecurrence of Cartan's curvature tensor. These spaces are important in the study of differential geometry due to their potential applications in both theoretical and applied mathematics.

In this paper, we emphasize the behavior of the curvature tensor under h-covariant derivatives and study the necessary and sufficient conditions that define generalized P^h-recurrent and P^h -birecurrent spaces. We establish numerous significant facts about the structure of these spaces and get explicit formulas for higher-order covariant derivatives of the curvature tensor. The P-Ricci tensor and vector tensor are also investigated in our research, providing additional information on their vanishing conditions and their connection to the geometry of these spaces. These results advance our knowledge of affine connections and their geometric applications in a variety of domains.

Specifically, new theoretical frameworks and applications have been made possible by the study of Cartan's second curvature tensor and its generalizations in recurrent Finsler spaces. Our knowledge of the geometrical structures involved in Finsler spaces has grown as a result of recent studies like those of Al-Qashbari (2020, 2023) and the investigation of generalized curvature tensors in these spaces (Qasem and Al-Qashbari, 2016). Studies like those of Pandey et al. (2011) and Hadi (2016), which concentrate on generalized recurrent spaces and their applications, further demonstrate the continued interest in the connections between curvature, torsion, and the recurrence qualities of these tensors. Recent studies have continued to advance the understanding of curvature structures in Finsler geometry by exploring refined properties and decompositions of fundamental tensors. Abdallah, Navlekar, and Ghadle (2022) investigated decomposition techniques for Cartan's second curvature tensor of various orders, providing new insights into the structural behavior of higher-order geometric objects in Finsler spaces. In parallel, Li (2022) introduced a Schur-type lemma concerning the mean Berwald curvature, which offers a deeper perspective on scalar curvature relations and isotropy in Finsler geometry. Furthermore, Sevim, Shen, and Ulgen (2023) contributed to this field by analyzing specific Ricci curvature tensors, enriching the theoretical framework with novel results that connect curvature properties to broader geometric structures. Together, these works reflect the continued expansion and sophistication of curvature analysis in modern Finsler geometry.

By examining the necessary and sufficient criteria for a space to be considered a generalized P^h-recurrent or P^h-birecurrent space, this study seeks to add to the body of knowledge already available on generalized third recurrent affinely connected spaces. The study expands on earlier research that examined the structure of generalized Finsler spaces with particular recurrence qualities, such as that done by Al-Qashbari and Baleedi (2023). We concentrate on the relationship between the geometric features of these spaces and the derivation of higher-order covariant derivatives of the curvature tensors. This investigation will advance the theoretical underpinnings of Finsler geometry and its applications in numerous scientific domains while also advancing our understanding of the behavior of curvature tensors in generalized recurrent spaces.

In the next sections, we will examine the recurrence conditions of Cartan's second curvature tensor, present the generalized third recurrent space, and formulate significant theorems concerning the structure of these spaces. The findings will contribute to the current discussion in differential geometry and its applications by offering fresh perspectives on the circumstances under which specific curvature components vanish, the function of h-covariant derivatives, and the connections among distinct geometric tensors.

The following equation describes the P^h-recurrent space:

$P^i_{jkh|\ell} = \lambda_\ell P^i_{jkh} + \mu_\ell(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $P^i_{jkh} \neq 0$, (1.1)

where λ_ℓ is the recurrence vector field, a non-zero covariant vector field.

We then investigated the idea of P^h-birecurrent spaces, which are defined by the following equation:

$P^i_{jkh|\ell|m} = a_{\ell m} P^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $P^i_{jkh} \neq 0$, (1.2)

In this case, the birecurrence tensor fields, $a_{\ell m}$ and $b_{\ell m}$, are non-zero covariant tensor fields of second order.

Covariant constants are the tensor g_{kh} and its associate tensor g^{kh} .

a) $g_{kh|r} = 0$ and b) $g^{kh}_{|r} = 0$. (1.3)

$g_{kr}\, g^{rh} = \delta^h_k = \begin{cases} 1 & \text{if } k = h , \\ 0 & \text{if } k \neq h . \end{cases}$ (1.4)

Both vector's y^i and y_i covariant derivatives vanish in the same way, that is,

a) $y^i_{|k} = 0$ and b) $y_{i|k} = 0$. (1.5)

a) $y_i\, y^i = F^2$ and b) $g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$. (1.6)

The following is satisfied by the vectors y_i and δ^i_k .

a) $\delta^i_k\, y^k = y^i$ and b) $\delta^i_k\, y_i = y_k$. (1.7)

a) $\delta^i_j\, g^{jk} = g^{ik}$ and b) $\delta^i_k\, \delta^k_h = \delta^i_h$. (1.8)

a) $\delta^i_k\, g_{ji} = g_{jk}$ and b) $g_{jh}\, y^j = y_h$. (1.9)

Using Euler, son homogeneous properties, this tensor satisfies the identities

a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$. (1.10)

In the directional argument, the hv-curvature tensor, also known as Cartan's second curvature tensor, is positively homogeneous of degree zero and is described by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm} P^m_{kh} - C^i_{jh|k}$$

or equivalent by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k}$$

or

$$P^i_{jkh} = C^i_{kh|j} - g^{ir}\, C_{jkh|r} + C^r_{jk}\, P^i_{rh} - P^r_{jh}\, C^i_{rk} .$$

[11] is satisfied by the hv-curvature tensor P^i_{jkh} , its related curvature tensor P_{ijkh}, the v(hv)-torsion tensor P^i_{kh} , the P-Ricci tensor P_{jk} , the division tensor P^i_k , and the vector tensor P_k .

a) $P^i_{jkh} y^j = P^i_{kh}$, b) $g_{ir} P^r_{jkh} = P_{ijkh}$, c) $P^i_{jki} = P_{jk}$, d) $P_{jk} g^{ji} = P^i_k$,

e) $P^i_{hk}\, y^h = 0$, f) $P^i_{ki} = P_k$, g) $P_{rjkh}\, g^{ri} = P^i_{jkh}$ and h) $P_{jk} g^{jk} = P$. (1.11)

a) $P^i_{jkh} = R^i_{jkh} - \frac{1}{3}(\delta^i_h R_{jk} - \delta^i_k R_{jh})$, b) $R^i_{jkh} = K^i_{jkh} + C^i_{jr}\, H^r_{hk}$,

c) $H^i_{jkh} = K^i_{jkh} + y^s(\dot{\partial}_j K^i_{skh})$. (1.12)

a) $R^i_{jkh} y^j = H^i_{kh} = K^i_{jkh}\, y^j$, b) $g_{ir} R^r_{jkh} = R_{ijkh}$, c) $R^i_{jki} = R_{jk}$, d) $R_{jk} g^{jk} = R$,

e) $g_{ir} K^r_{jkh} = K_{ijkh}$, f) $K^i_{jki} = K_{jk}$, g) $K_{jk} g^{jk} = K$,

h) $H^i_{kh} y^k = H^i_h$,

i) $H = \frac{1}{(n-1)} H^i_i$ and j) $H^i_{ki} = H_k$. (1.13)

II. Cartan's Second Curvature Tensor in Generalized P^h-Recurrent Spaces: Generalized Recurrence Conditions and Birecurrence Relations

Cartan's curvature tensors are essential to the study of differential geometry in comprehending manifolds' inherent geometry. This study examines the characteristics of The second curvature tensor of Cartan, represented by the symbol P^i_{jkh} when discussing generalized P^h-recurrent spaces. We investigate the birecurrence and extended recurrence conditions that control this tensor's behavior in spaces with non-zero covariant vector fields. In particular, we examine how the tensor behaves under successive covariant differentiations and how it relates to its covariant derivatives. We define generalized P^h-recurrent spaces and extend them to birecurrent and trirecurrent spaces by studying generalized recurrence requirements for the second curvature tensor. We offer a thorough examination of these recurrence requirements in a variety of geometric contexts and develop important relations that characterize the behavior of the curvature tensor under higher-order covariant derivatives. In addition to adding to the existing knowledge of Cartan's curvature tensors, the results provided here further advance the theory of recurrent spaces in differential geometry. By investigating the curvature tensor in these generalized recurrent spaces, we gain fresh insights into the manifolds' structure and geometric features. Important theorems and conditions that give necessary and sufficient criteria for the existence of such spaces are also included in the study, enhancing the current theoretical framework in the topic.

The generalized recurrence requirement was satisfied by Cartan's second curvature tensor P^i_{jkh} .

$P^i_{jkh|l} = \alpha_l P^i_{jkh} + \beta_l(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \frac{1}{4}\delta_l(P^i_k\, g_{jh} - P^i_h\, g_{jk})$, $P^i_{jkh} \neq 0$, (2.1)

In this case, $|l$ is the h-covariant derivative of order one with respect to x^l , α_l , β_l and δ_l , which are non-zero covariant vector fields. The space is referred to as a generalized P^h-recurrent space.

In Cartan, the generalized birecurrence requirement was satisfied by the second curvature tensor

$P^i_{jkh|l|m} = a_{lm} P^i_{jkh} + b_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \frac{1}{4} c_{lm}(P^i_k\, g_{jh} - P^i_h g_{jk})$
$+ \frac{1}{4}\delta_l\left(P^i_{k|m}\, g_{jh} - P^i_{h|m}\, g_{jk}\right)$, $P^i_{jkh} \neq 0$, (2.2)

where $|l|m$ is h-covariant derivative of order two with respect to x^l and x^m, successively, α_{lm} , β_{lm} , c_{lm} , $a_{lm} = \alpha_{l|m} + \alpha_l \alpha_m$, $b_{lm} = \beta_{l|m} + \alpha_l \beta_m$ and $c_{lm} = \frac{1}{4}(\delta_{l|m} + \alpha_l \delta_m)$ are non-zero covariant vectors field and the space is called a generalized P^h-birecurrent space.

Using condition (1.5a) and the h-covariant derivative of (2.2), with regard to x^n, we obtain

$$P^i_{jkh|l|m|n} = \lambda_{lmn} P^i_{jkh} + \mu_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\gamma_{lmn}(P^i_k g_{jh} - P^i_h g_{jk}) + \frac{1}{4}c_{lm}\left(P^i_{k|n} g_{jh} - P^i_{h|n} g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} g_{jh} - P^i_{h|m} g_{jk}\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{jh} - P^i_{h|m|n} g_{jk}\right) ,$$

$$P^i_{jkh} \neq 0 . \tag{2.3}$$

where $|l|m|n$ is h-covariant derivative of order three with respect to x^l, x^m and x^n successfully, $\lambda_{lmn} = a_{lm|n} + a_{lm}\alpha_n$, $\mu_{lmn} = b_{lm|n} + a_{lm}\beta_n$ and $\gamma_{lmn} = \frac{1}{4}(c_{lm|n} + a_{lm}\delta_n)$ are non-zero covariant tensors fields of order three.

P^h-generalized trirecurrent space is the space and tensor satisfying (2.3). For short, we'll refer to them as P^h-G-TRF$_n$.

A. *Result There is a generalized P^h-Trirecurrent space for every generalized P^h-Birrecurrent space.*

By utilizing (1.3a), (1.9a), and (1.11b) to transvect (2.3) by g_{ir} , we obtain

$$P_{jrkh|l|m|n} = \lambda_{lmn} P_{jrkh} + \mu_{lmn}(g_{kr} g_{jh} - g_{hr} g_{jk}) + \frac{1}{4}\gamma_{lmn}(P_{kr} g_{jh} - P_{hr} g_{jk}) + \frac{1}{4}c_{lm}(P_{kr|n} g_{jh} - P_{hr|n} g_{jk}) + \frac{1}{4}\delta_{l|n}(P_{kr|m} g_{jh} - P_{hr|m} g_{jk}) + \frac{1}{4}\delta_l(P_{kr|m|n} g_{jh} - P_{hr|m|n} g_{jk}) . \tag{2.4}$$

Conversely, transvection (2.4) by g^{ir}, using (1.3b), (1.4) and (1.11g), yields condition (2.3).

The proof of theorem is completed, we conclude

B. *Theorem The space P^h-G-TRF$_n$ may characterized by the condition (2.4).*

Transvecting (2.3) by y^j , using (1.5a), (1.9b) and (1.11a), we get

$$P^i_{kh|l|m|n} = \lambda_{lmn} P^i_{kh} + \mu_{lmn}(\delta^i_k y_h - \delta^i_h y_k) + \frac{1}{4}\gamma_{lmn}(P^i_k y_h - P^i_h y_k) + \frac{1}{4}c_{lm}\left(P^i_{k|n} y_h - P^i_{h|n} y_k\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} y_h - P^i_{h|m} y_k\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} y_h - P^i_{h|m|n} y_k\right) . \tag{2.5}$$

From the previous steps, we can conclude the following theorem

C. *Theorem In P^h-G-TRF$_n$, the h-covariant derivative of third order for the v(hv)-torsion tensor P^i_{kh} is given by (2.5).*

Contracting the indices i and h in (2.3) and (2.5), using (1.11c) and (1.11f), we get

$$P_{jk|l|m|n} = \lambda_{lmn} P_{jk} + (1-n)\mu_{lmn} g_{jk} + \frac{1}{4}\gamma_{lmn}(P^i_k g_{ji} - P g_{jk}) + \frac{1}{4}c_{lm}\left(P^i_{k|n} g_{ji} - P_{|n} g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} g_{ji} - P_{|m} g_{jk}\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{ji} - P_{|m|n} g_{jk}\right). \tag{2.7}$$

$$P_{k|l|m|n} = \lambda_{lmn} P_k + \mu_{lmn}(1-n) y_k + \frac{1}{4}\gamma_{lmn}(P^i_k y_i - P y_k) + \frac{1}{4}c_{lm}\left(P^i_{k|n} y_i - P_{|n} y_k\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} y_i - P_{|m} y_k\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} y_i - P_{|m|n} y_k\right) . \tag{2.8}$$

Transvecting (2.7) by g^{jk} , using (1.3b), (1.4) and (1.11h), we get

$$P_{|l|m|n} = \lambda_{lmn} P + (1-n)\mu_{lmn} . \tag{2.9}$$

The following theorem can be inferred from the earlier steps.

D. *Theorem In P^h-G-TRF$_n$, the P-Ricci tensor P_{jk} , the vector tensor P_k and the scalar curvature P are non-vanishing.*

III. Relations Between Cartan's Second Curvature Tensor P^i_{jkh} and Other Tensors of Curvature

The curvature tensor is essential to comprehending the geometric characteristics of a manifold in differential geometry. P^i_{jkh} , Cartan's second curvature tensor, is an essential tool for studying curvature in a variety of geometric situations, especially recurrent spaces. The behavior of the curvature in these spaces, which display particular recurrence features under covariant differentiation, requires this tensor. To advance the theory of generalized recurrent spaces, it is essential to comprehend the relationships between Cartan's second curvature tensor and other tensors, including the Riemann curvature tensor, the Ricci tensor, and other higher-order curvature quantities. This section's main goal is to examine the relationships between Cartan's second curvature tensor and several other curvature tensors, with a focus on how the geometry of the underlying space affects these relationships. We give a deeper insight into the interaction of Cartan's tensor with other geometric objects in the context of generalized P^h-recurrent spaces by examining the covariant derivatives and recurrence conditions.

In order to develop a cohesive framework for researching the curvature behavior in spaces with higher-dimensional and recurrent structures, these interactions are essential. Through in-depth calculations, we examine the tensor relations and shed light on how these tensors behave under covariant differentiation. With applications in theoretical physics and geometry, the findings should improve our knowledge of the function of Cartan's second curvature tensor in higher-order differential geometry.

Taking the h-covariant derivative of third order for the formula (1.12a), with respect to x^l, x^m and x^n , successively and using (2.3), we get

$$R^i_{jkh|l|m|n} = \lambda_{lmn} R^i_{jkh} + \mu_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\gamma_{lmn}(P^i_k g_{jh} - P^i_h g_{jk}) + \frac{1}{4}c_{lm}\left(P^i_{k|n} g_{jh} - P^i_{h|n} g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} g_{jh} - P^i_{h|m} g_{jk}\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{jh} - P^i_{h|m|n} g_{jk}\right) + \frac{1}{3}(\delta^i_k R_{jh} - \delta^i_h R_{jk})_{|l|m|n} - \frac{1}{3}\lambda_{lmn}(\delta^i_k R_{jh} - \delta^i_h R_{jk}) . \tag{3.1}$$

Which can write as

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$R^i_{jkh|l|m|n} = \lambda_{lmn} R^i_{jkh} + \mu_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_{lmn}\left(P^i_k g_{jh} - P^i_h\, g_{jk}\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n} g_{jh} - P^i_{h|n} g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} g_{jh} - P^i_{h|m} g_{jk}\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{jh} - P^i_{h|m|n} g_{jk}\right)$. (3.2)

If and only if

$(\delta^i_h R_{jk} - \delta^i_k R_{jh})_{|l|m|n} = \lambda_{lmn}\,(\delta^i_h R_{jk} - \delta^i_k R_{jh})$. (3.3)

From the previous steps, we can conclude the following theorem

A. Theorem In P^h-G-TRF$_n$, Cartan's third curvature tensor R^i_{jkh} is generalized trirecurrent if and only if equation (3.3) holds good.

Transvecting (3.1) by g_{ir} , using (1.3a), (1.9a) and (1.13b), we get

$R_{jrkh|l|m|n} = \lambda_{lmn} R_{jrkh} + \mu_{lmn}\left(g_{kr} g_{jh} - g_{hr} g_{jk}\right) + \frac{1}{4}\gamma_{lmn}\left(P_{kr} g_{jh} - P_{hr} g_{jk}\right)$
$+ \frac{1}{4}c_{lm}\left(P_{kr|n}\, g_{jh} - P_{hr|n}\, g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P_{kr|m}\, g_{jh} - P_{hr|m}\, g_{jk}\right)$
$+ \frac{1}{4}\delta_l\left(P_{kr|m|n}\, g_{jh} - P_{hr|m|n}\, g_{jk}\right)$. (3.4)

If and only if

$(g_{hr} R_{jk} - g_{kr} R_{jh})_{|l|m|n} = \lambda_{lmn}\,(g_{hr} R_{jk} - g_{kr} R_{jh})$ (3.5)

From the previous steps, we can conclude the following theorem

B. *Theorem* In P^h-G-TRF$_n$, the associate curvature tensor R_{jrkh} of Cartan's third curvature tensor R^i_{jkh} is generalized trirecurrent if and only if equation (3.5) holds good.

Transvecting (3.1) by y^j , using (1.5a), (1.9b) and (1.13a), we get

$H^i_{kh|l|m|n} = \lambda_{lmn} H^i_{kh} + \mu_{lmn}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{4}\gamma_{lmn}\left(P^i_k\, y_h - P^i_h\, y_k\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n} y_h - P^i_{h|n} y_k\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m}\, y_h - P^i_{h|m} y_k\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n}\, y_h - P^i_{h|m|n} y_k\right)$. (3.6)

If and only if

$\left(\delta^i_h R_k - \delta^i_k R_h\right)_{|l|m|n} = \lambda_{lmn}\left(\delta^i_h R_k - \delta^i_k R_h\right)$. (3.7)

Using (1.13j) to contract the indices i and h in (3.6) and (3.7), we obtain

$H_{k|l|m|n} = \lambda_{lmn} H_k + \mu_{lmn}(1-n) y_k + \frac{1}{4}\gamma_{lmn}\left(P^i_k\, y_i - P y_k\right) + \frac{1}{4}c_{lm}\left(P^i_{k|n} y_i - P_{|n} y_k\right)$
$+\frac{1}{4}\delta_{l|n}\left(P^i_{k|m}\, y_i - P_{|m} y_k\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n}\, y_i - P_{|m|n} y_k\right)$. (3.8)

If and only if

$R_{k|l|m|n} = \lambda_{lmn} R_k$. (3.9)

Transvecting (3.6) and (3.7) by y^k , using (1.5a), (1.9b) and (1.13h), we get

$H^i_{h|l|m|n} = \lambda_{lmn} H^i_h + \mu_{lmn}\left(y^i y_h - \delta^i_h F^2\right) + \frac{1}{4}\gamma_{lmn}\left(P^i_k\, y_h y^k - P^i_h\, F^2\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n} y_h y^k - P^i_{h|n} F^2\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} y_h y^k - P^i_{h|m} F^2\right)$
$+\frac{1}{4}\delta_l\left(P^i_{k|m|n}\, y_h y^k - P^i_{h|m|n} F^2\right)$. (3.10)

If and only if

$\left(\delta^i_h R - y^i R_h\right)_{|l|m|n} = \lambda_{lmn}\left(\delta^i_h R - y^i R_h\right)$. (3.11)

Contracting the indices i and h in (3.10) and (3.11), using (1.13i), we get

$H_{|l|m|n} = (n-1)\lambda_{lmn} H - \mu_{lmn} F^2 + \frac{1}{4(n-1)}\gamma_{lmn}\left(P^i_k\, y_i y^k - P F^2\right)$
$+ \frac{1}{4(n-1)} c_{lm}\left(P^i_{k|n} y_i y^k - P_{|n} F^2\right) + \frac{1}{4(n-1)}\delta_{l|n}\left(P^i_{k|m}\, y_i y^k - P_{|m} F^2\right)$
$+\frac{1}{4(n-1)}\delta_l\left(P^i_{k|m|n}\, y_i y^k - P_{|m|n} F^2\right)$. (3.12)

If and only if

$R_{|l|m|n} = \lambda_{lmn} R$. (3.13)

From the previous steps, we can conclude the following theorem

C. *Theorem In P^h-G-TRF$_n$, the h-covariant derivative of third order for the $h(v)$-torsion tensor H^i_{kh} , the deviation tensor H^i_h , the curvature vector H_k and the scalar curvature H are non- vanishing.*

Contracting the indices i and h in (3.1), using (1.13c), we get

$R_{jk|l|m|n} = \lambda_{lmn} R_{jk} + (1-n)\mu_{lmn}\, g_{jk} + \frac{1}{4}\gamma_{lmn}\left(P^i_k g_{ji} - P\, g_{jk}\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n}\, g_{ji} - P_{|n}\, g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m}\, g_{ji} - P_{|m}\, g_{jk}\right)$
$+\frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{ji} - P_{|m|n} g_{jk}\right)$. (3.14)

If and only if

$R_{jk\,|l|m|n} = \lambda_{lmn} R_{jk}$. (3.15)

Transvecting (3.14) and (3.15) by g^{jk} , using (1.3b), (1.4) and (1.13d), we get

$R_{|l|m|n} = \lambda_{lmn} R + (1-n)\mu_{lmn}$. (3.16)

If and only if

$R_{|l|m|n} = \lambda_{lmn} R$. (3.17)

From the previous steps, we can conclude the following theorem

D. Theorem 3.4. In P^h-G-TRF$_n$, the R-Ricci tensor R_{jk} , and the scalar curvature tensor R are non- vanishing.

Taking the h-covariant derivative of third order for the formula (1.12b), with respect to x^l, x^m and x^n , successively and using (3.1), we get

$K^i_{jkh|l|m|n} = \lambda_{lmn} K^i_{jkh} + \mu_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_{lmn}\left(P^i_k g_{jh} - P^i_h\, g_{jk}\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n} g_{jh} - P^i_{h|n} g_{jk}\right) + \frac{1}{4}\delta_{l|n}\left(P^i_{k|m} g_{jh} - P^i_{h|m}\, g_{jk}\right) + \frac{1}{4}\delta_l\left(P^i_{k|m|n} g_{jh} - P^i_{h|m|n} g_{jk}\right)$
$+\frac{1}{3}\left(\delta^i_k R_{jh} - \delta^i_h R_{jk}\right)_{|l|m|n} - \frac{1}{3}\lambda_{lmn}\left(\delta^i_k R_{jh} - \delta^i_h R_{jk}\right) + \left(C^i_{jr}\, H^r_{hk}\right)_{|l|m|n} - \lambda_{lmn}\left(C^i_{jr}\, H^r_{hk}\right)$. (3.18)

Which can written as

$K^i_{jkh|l|m|n} = \lambda_{lmn} K^i_{jkh} + \mu_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_{lmn}\left(P^i_k g_{jh} - P^i_h\, g_{jk}\right)$

$+\frac{1}{4}c_{lm}\left(P^i_{k|n}g_{jh}-P^i_{h|n}g_{jk}\right)+\frac{1}{4}\delta_{l|n}\left(P^i_{k|m}g_{jh}-P^i_{h|m}g_{jk}\right)+\frac{1}{4}\delta_l\left(P^i_{k|m|n}g_{jh}-P^i_{h|m|n}g_{jk}\right)$. (3.19)

If and only if

$\frac{1}{3}\left(\delta^i_hR_{jk}-\delta^i_kR_{jh}\right)_{|l|m|n}+\left(C^i_{jr}H^r_{hk}\right)_{|l|m|n}=\frac{1}{3}\lambda_{lmn}\left(\delta^i_hR_{jk}-\delta^i_kR_{jh}\right)+\lambda_{lmn}\left(C^i_{jr}H^r_{hk}\right)$. (3.20)

From the previous steps, we can conclude the following theorem

E. Theorem In P^h-G-TR F_n , Cartan's fourth curvature tensor K^i_{jkh} is generalized trirecurrent if and only if equation (3.20) holds well.

Transvecting (3.18) by g_{ir} , using (1.3a), (1.9a) and (1.13e), we get

$K_{jrkh|l|m|n}=\lambda_{lmn}K_{jrkh}+\mu_{lmn}\left(g_{kr}g_{jh}-g_{hr}g_{jk}\right)+\frac{1}{4}\gamma_{lmn}\left(P_{kr}g_{jh}-P_{hr}g_{jk}\right)$
$+\frac{1}{4}c_{lm}\left(P_{kr|n}\,g_{jh}-P_{hr|n}\,g_{jk}\right)+\frac{1}{4}\delta_{l|n}\left(P_{kr|m}\,g_{jh}-P_{hr|m}\,g_{jk}\right)$
$+\frac{1}{4}\delta_l\left(P_{kr|m|n}\,g_{jh}-P_{hr|m|n}\,g_{jk}\right)$. (3.21)

If and only if

$\frac{1}{3}\left(g_{hr}R_{jk}-g_{kr}R_{jh}\right)_{|l|m|n}+\left(C^i_{jr}\,H_{ihk}\right)_{|l|m|n}$
$=\frac{1}{3}\lambda_{lmn}\left(g_{hr}R_{jk}-g_{kr}R_{j3.2.h}\right)+\lambda_{lmn}\left(C^i_{jr}\,H_{ihk}\right)$
. (3.22)

From the previous steps, we can conclude the following theorem

F. *Theorem* In P^h-G-TRF_n , the associate curvature tensor K_{jrkh} of Cartan's fourth curvature tensor K^i_{jkh} is generalized trirecurrent if and only if equation (3.22) holds good.

Contracting the indices i and h in (3.18), using (1.13f), we get

$K_{jk|l|m|n}=\lambda_{lmn}K_{jk}+(1-n)\,\mu_{lmn}\,g_{jk}+\frac{1}{4}\gamma_{lmn}\left(P^i_kg_{ji}-P\,g_{jk}\right)$
$+\frac{1}{4}c_{lm}\left(P^i_{k|n}\,g_{ji}-P_{|n}\,g_{jk}\right)+\frac{1}{4}\delta_{l|n}\left(P^i_{k|m}\,g_{ji}-P_{|m}\,g_{jk}\right)$
$+\frac{1}{4}\delta_l\left(P^i_{k|m|n}\,g_{ji}-P_{|m|n}\,g_{jk}\right)$. (3.23)

If and only if

$\frac{1}{3}R_{jk}{}_{|l|m|n}+\left(C^i_{jr}\,H^r_{hk}\right)_{|l|m|n}=\frac{1}{3}\lambda_{lmn}R_{jk}+\lambda_{lmn}C^i_{jr}\,H^r_{hk}$.
. (3.24)

Transvecting (3.23) and (3.24) by g^{jk} , using (1.3b), (1.4) and (1.13g), we get

$K_{|l|m|n}=\lambda_{lmn}K+(1-n)\mu_{lmn}$. (3.25)

If and only if

$\frac{1}{3}R_{|l|m|n}+(C^i_r\,H^r_h)_{|l|m|n}=\lambda_{lmn}\left(\frac{1}{3}R+C^i_r\,H^r_h\right)$. (3.26)

From the previous steps, we can conclude the following theorem

G. Theorem In P^h-G-TRF_n , the K-Ricci tensor K_{jk} , and the scalar curvature tensor K are non- vanishing.

IV. Recommendations

Based on the detailed findings and analyses presented in this paper, the following recommendations can be made for further research and development in the field of differential geometry, particularly in the study of Cartan's second curvature tensor and its recurrence conditions: **Further Exploration of Higher-Order Derivatives**: The investigation of higher-order covariant derivatives of the curvature tensors, as discussed in the generalized recurrence and birecurrence conditions, is essential for a more comprehensive understanding of curvature behavior in generalized P^h-recurrent spaces. Future work should focus on extending these conditions to higher dimensions and more complex recurrent geometries, enabling the development of generalized curvature models that can be applied in both mathematical and physical contexts.

Numerical Simulations for Practical Applications: The abstract mathematical models presented in this paper can be further developed through numerical simulations, especially in the context of physical theories where curvature plays a pivotal role, such as in general relativity and cosmology. The derived recurrence and birecurrence conditions could be tested against real-world data to evaluate their relevance and accuracy in describing the curvature of manifolds with complex structures.

Generalization to Non-Recurrent Geometries: While the focus of this paper is on generalized P^h-recurrent spaces, it is crucial to examine the implications of these recurrence conditions in non-recurrent geometries. Future research can investigate whether the relationships between Cartan's second curvature tensor and other curvature tensors hold in non-recurrent spaces, providing new insights into the broader class of geometric manifolds.

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