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A Study of the M-Projective Curvature Tensor \overline{W}^i_{jkh} in Generalized Recurrent and Birecurrent Finsler Spaces

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Abstract— This paper aims to examine the properties of the M-projective curvature tensor in the context of generalized Finsler spaces, specifically within the framework of a \mathcal{B} -space. The study begins with the derivation of the M-projective curvature tensor, which is expressed as the sum of the standard M-projective curvature tensor and additional terms involving the Ricci tensor and scalar curvature. Through covariant differentiation, the behavior of this tensor under certain conditions is analyzed, leading to a set of conditions necessary for the space to exhibit generalized recurrent Finsler properties. The paper includes multiple theorems that explore these relationships, proving that the M-projective curvature tensor satisfies the conditions for generalized recurrent Finsler spaces. Additionally, the work introduces the concept of generalized birecurrent Finsler spaces and presents several characterization theorems. Finally, the results are corroborated with computational formulations that demonstrate the conditions under which the tensor relationships hold true.

Keywords— The \mathcal{B} -covariant derivative of first and second orders, Generalized recurrent and birecurrent Finsler space, Weyl tensor W^i_{jkh} and the M-projective curvature tensor \overline{W}^i_{jkh} .

I. INTRODUCTION

The study of curvature tensors and their applications in Finsler geometry has been a subject of considerable interest due to their importance in understanding the geometric properties of spaces. The M-projective curvature tensor, denoted \overline{W}^i_{jkh} , is central to this paper, and its study within the context of \mathcal{B} -spaces provides new insights into the geometric structure of generalized Finsler spaces. These spaces are an extension of classical Finsler geometry, incorporating more general forms of curvature and torsion.

The main objective of this work is to investigate the properties of the M-projective curvature tensor and establish its connection to generalized recurrent Finsler spaces. A

series of equations is derived that detail the behavior of the tensor under covariant derivatives, leading to several key results. These results are encapsulated in the form of theorems that provide necessary and sufficient conditions for the M-projective curvature tensor to define a generalized recurrent Finsler space. Additionally, we introduce and explore the concept of generalized birecurrent Finsler spaces, extending the classical notion of birecurrence to a more general context. The work culminates in the formulation of several important conditions that must be satisfied for the tensor to adhere to the structure of generalized Finsler spaces, thereby enriching the understanding of the geometric properties of these spaces.

The theoretical foundation laid in this study paves the way for future research into more complex geometric structures, where such tensors play a crucial role in understanding the curvature and torsion of higher-dimensional spaces. The results presented in this paper have potential applications in theoretical physics, particularly in the study of spacetime geometries and general relativity, as well as in differential geometry and mathematical physics.

The study of curvature tensors in Finsler geometry has garnered significant attention over the years, with several key contributions advancing our understanding of these geometric structures. One notable area of research focuses on the properties and applications of the w^* -curvature tensor in relativistic spacetimes. Abu-Donia, Shenawy, and Abdehameed (2020) [1] investigate the w^* -curvature tensor, shedding light on its role in relativistic geometries, which serves as a foundation for exploring more general tensorial structures in Finsler manifolds. This study has influenced subsequent work on curvature tensors in both general relativity and specialized geometric spaces.

Ahsan and Ali (2014, 2016) [2, 3] provide a detailed analysis of the w -curvature tensor, introducing fundamental properties and implications for spacetime in the framework of general relativity. Their work has been pivotal in understanding the behavior of curvature tensors within both classical and

relativistic contexts and has been further extended by researchers working on higher-order curvature structures.

Al-Qashbari et al. (2025) [4, 5] have made significant strides in the study of M-projective curvature tensors, focusing on their decomposition and the role of Lie derivatives in generalized Finsler spaces. Their contributions offer critical insights into higher-order tensors, providing an extensive analysis of the M-projective curvature tensor and its transvecting forms in the context of Finsler geometry. This has provided a rigorous framework for subsequent studies on generalizations of Finsler spaces, especially in terms of torsion and curvature structures.

The work of Al-Qashbari, Abdallah, and their collaborators (2024, 2025) [6, 7, 8] explores generalized recurrent Finsler spaces, utilizing both Berwald's and Cartan's covariant derivatives to define and study higher-order curvature tensors. These studies examine the interplay between torsion and curvature, further enriching the theoretical foundations of Finsler spaces, particularly in relation to the geometric structures defined by Cartan's covariant derivatives.

Further contributions by Misra et al. (2014) [9] and Goswami (2017) [10] have examined recurrent Finsler spaces with special curvature tensors, deepening the understanding of higher-order geometric structures in these spaces. Misra et al. analyze recurrent Finsler spaces with Berwald's curvature tensor field, while Goswami's systematic review provides a comprehensive overview of special Finsler spaces and their differential geometric properties. These works collectively provide a broad perspective on the evolution of curvature tensor research, especially in the context of Finsler spaces. Pandey, Saxena, and Goswami (2011) [11] focus on the generalized H-recurrent spaces, offering an in-depth exploration of their curvature tensors and their application in various geometric settings. This work complements the broader study of recurrent Finsler spaces, laying the groundwork for further investigations into generalized curvature structures and their implications in mathematical physics. Rund (1981) [12] presents a classical text on the differential geometry of Finsler spaces, offering foundational principles that underpin much of the modern study in this field. His work has influenced numerous subsequent studies, including those investigating the advanced properties of higher-order curvature tensors and their applications in both pure mathematics and theoretical physics.

In this study, several identities are explored to establish a relationship between Weyl's curvature tensor and the M-projective curvature tensor. Initially, a comprehensive introduction is provided to the foundational concepts of both Weyl's curvature tensor and the M-projective curvature tensor. Subsequently, a set of identities is driven that elucidate the connection between these two tensorial structures, contributing to the broader understanding of their geometric properties.

II. PRELIMINARIES

In this section, we establish several key conditions and definitions crucial for the development of the paper. The metric tensor and Berwald's connection coefficients are both positively homogeneous of degree 0 with respect to the directional arguments.

Consider two vectors y_i and y^i that satisfy the following relations

$$\begin{aligned} \text{a) } & y_i = g_{ij} y^j, \quad \text{b) } y_i y^i = F^2 \quad \text{and} \\ \text{c) } & \delta_j^k y^j = y^k. \end{aligned} \quad (2.1)$$

The metric tensors g_{ij} and g^{ij} are connected by the identity:

$$\begin{aligned} \text{a) } & g_{ij} g^{jk} = \delta_i^k = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}, \\ \text{b) } & g_{ir} \delta_j^i = g_{rj} \quad \text{and} \quad \text{c) } g^{jk} \delta_k^i = g^{ji}. \end{aligned} \quad (2.2)$$

The torsion tensor C_{ijk} satisfies the condition:

$$C_{ijk} y^i = C_{ijk} y^j = C_{ijk} y^k = 0. \quad (2.3)$$

The covariant derivative $\mathcal{B}_k T_j^i$ of any tensor T_j^i with respect to Berwald's connection is defined as

$$\mathcal{B}_k T_j^i = \partial_k T_j^i - (\partial_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r. \quad (2.4)$$

Additionally, the vector y^i and the metric function F vanish identically under Berwald's covariant derivative:

$$\text{a) } \mathcal{B}_k F = 0 \quad \text{and} \quad \text{b) } \mathcal{B}_k y^i = 0. \quad (2.5)$$

The metric tensor g_{ij} is non-zero and satisfies

$$\mathcal{B}_k g_{ij} = -2 C_{ijk} y^h = -2 y^h \mathcal{B}_h C_{ijk}. \quad (2.6)$$

The curvature tensor W_{jkh}^i , torsion tensor W_{jk}^i and deviation tensor W_j^i are defined as follows:

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2 \delta_j^i}{(n+1)} H_{[hk]} + \frac{2 y^i}{(n+1)} \partial_j H_{[kh]} \\ &+ \frac{\delta_k^i}{(n^2-1)} (n H_{jh} + H_{hj} + y^r \partial_j H_{hr} \\ &- \frac{\delta_h^i}{(n^2-1)} (n H_{jk} + H_{kj} + y^r \partial_j H_{kr})), \end{aligned} \quad (2.7)$$

$$\begin{aligned} W_{jk}^i &= H_{jk}^i + \frac{y^i}{(n+1)} H_{[jk]} \\ &+ 2 \left\{ \frac{\delta_{[j}^i}{(n^2-1)} (n H_{k]} - y^r H_{k]r} \right\}, \end{aligned} \quad (2.8)$$

$$W_j^i = H_j^i - H \delta_j^i - \frac{1}{(n+1)} (\partial_r H_j^r - \partial_j H) y^i, \quad (2.9)$$

respectively.

Moreover, we assume that the tensor W_j^i satisfies the following identities:

$$\begin{aligned} \text{a) } & W_k^i y^k = 0, \quad \text{b) } W_i^i = 0, \quad \text{c) } W_k^i y_i = 0, \\ \text{d) } & g_{ir} W_j^i = W_{rj}, \quad \text{e) } g^{jk} W_{jk} = W \quad \text{and} \\ \text{f) } & W_{jk} y^k = 0. \end{aligned} \quad (2.10)$$

The M-projective curvature tensor \bar{W}_{jkh}^i , torsion tensor \bar{W}_{kh}^i , Ricci tensor \bar{W}_{jk} , curvature vector \bar{W}_k and scalar curvature \bar{W} satisfy the following relations:

$$\begin{aligned} \text{a) } & \bar{W}_{jkh}^i y^j = \bar{W}_{kh}^i, \quad \text{b) } \bar{W}_{kh}^i y^k = \bar{W}_h^i, \\ \text{c) } & \bar{W}_{jki}^i = \bar{W}_{jk}, \quad \text{d) } \bar{W}_{ki}^i = \bar{W}_k, \\ \text{e) } & \bar{W}_i^i = \bar{W} \quad \text{and} \quad \text{f) } g_{ir} \bar{W}_{jkh}^i = \bar{W}_{rjkh}. \end{aligned} \quad (2.11)$$

Cartan's third curvature tensor R_{jkh}^i , Ricci tensor R_{jk} , the vector H_k and scalar curvature H are defined as

$$\begin{aligned} \text{a) } & R_{jk} y^j = H_k, \quad \text{b) } R_{jk} y^k = R_j, \quad \text{c) } R_i^i = R, \\ \text{d) } & g_{ir} R_j^i = R_{rj} \quad \text{and} \quad \text{e) } H_k y^k = (n-1)H. \end{aligned} \quad (2.12)$$

Al-Qashbari and AL-ssallal [6], Al-Qashbari, Haouse and AL-ssallal [7] introduced and studied of curvature tensor by using Berwald's and Cartan's first and second order derivatives in Finsler space which characterized by the conditions:

$$\begin{aligned} \mathcal{B}_m W_{jkh}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}). \end{aligned} \quad (2.13)$$

A Finsler space F_n , in which the curvature tensor W_{jkh}^i satisfies the condition (2.13), is referred to as the generalized BW -recurrent space and denoted by $G^{2nd}BW - RF_n$. Taking the covariant derivative of (2.13) with respect to x^l in the context of Berwald's connection, we obtain:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= (\mathcal{B}_l \lambda_m) W_{jkh}^i + \lambda_m (\mathcal{B}_l W_{jkh}^i) \\ &+ (\mathcal{B}_l \mu_m) (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \mu_m \mathcal{B}_l (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_k^i) g_{jh} + W_k^i (\mathcal{B}_l g_{jh}) - (\mathcal{B}_l W_h^i) g_{jk} - \\ &W_h^i (\mathcal{B}_l g_{jk})) \\ &+ \frac{1}{4} (\mathcal{B}_l \gamma_m) (W_k^i g_{jh} - W_h^i g_{jk}). \end{aligned} \quad (2.14)$$

By applying equations (2.6) and (2.13) to equation (2.14), we get

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= \lambda_{ml} W_{jkh}^i + \lambda_m (\lambda_l W_{jkh}^i + \mu_l (\delta_k^i g_{jh} - \\ &\delta_h^i g_{jk}) + \frac{1}{4} \gamma_l (W_k^i g_{jh} - W_h^i g_{jk})) \\ &+ \mu_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) - 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \\ &\delta_h^i C_{jkl}) + \frac{1}{4} \gamma_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_k^i) g_{jh} - (\mathcal{B}_l W_h^i) g_{jk}) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}). \end{aligned}$$

Or

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= (\lambda_{ml} + \lambda_m \lambda_l) W_{jkh}^i \\ &+ (\mu_{ml} + \lambda_m \mu_l) (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_k^i) g_{jh} - (\mathcal{B}_l W_h^i) g_{jk}) \\ &+ \frac{1}{4} (\lambda_m \gamma_l + \gamma_{ml}) (W_k^i g_{jh} - W_h^i g_{jk}) \\ &- 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}). \end{aligned} \quad (2.15)$$

The equation (2.15), can be expressed as:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= a_{ml} W_{jkh}^i + b_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} c_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_k^i) g_{jh} - (\mathcal{B}_l W_h^i) g_{jk}) \\ &- 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}). \end{aligned} \quad (2.16)$$

Where $a_{ml} = \lambda_{ml} + \lambda_m \lambda_l$, $b_{ml} = \mu_{ml} + \lambda_m \mu_l$ and $c_{ml} = \lambda_m \gamma_l + \gamma_{ml}$ are non-zero second order covariant tensor fields, γ_m and μ_m are non-zero first order covariant vector fields, respectively.

Definition 2.1. In a Finsler space where the Weyl's projective curvature tensor W_{jkh}^i satisfies the condition (2.16), it is referred to as a generalized BW -birecurrent space. The tensor is called a generalized B -birecurrent tensor. These spaces and tensors are denoted as $G^{2nd}BW - BRF_n$ and $G^{2nd}B - BR$, respectively.

Characterization of M-Projective Curvature \bar{W}_{jkh}^i and Deviation Tensors in $G^{2nd}BW - RF_n$ Spaces and $G^{2nd}B\bar{W} - BRF_n$ Spaces

This paper focuses on the characterization of the M-projective curvature tensor \bar{W}_{jkh}^i and deviation tensors in the context of generalized Finsler spaces, specifically within the framework of $G^{2nd}B\bar{W} - RF_n$ spaces. We aim to explore the

geometric properties and structural relationships of these tensors, providing a comprehensive analysis of their roles in describing curvature and deviation phenomena in higher-dimensional Finsler geometries. The study contributes to the understanding of M-projective curvature in relation to generalized Finsler manifolds and offers insights into their potential applications in advanced differential geometry and theoretical physics.

Some properties of W_{jkh}^i curvature tensor was proposed by Al-Qashbari, Abdallah and Al-sallal.

For $(n = 4)$ a Riemannian space, the M-projective curvature tensor \bar{W}_{jkh}^i is given by

$$\begin{aligned} W_{jkh}^i &= \bar{W}_{jkh}^i + \frac{1}{6} (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &- \frac{1}{6} (g_{jh} R_k^i + g_{jk} R_h^i). \end{aligned} \quad (3.1)$$

By taking the covariant derivative of (3.1), with respect to x^m in the context of Berwald we get

$$\begin{aligned} \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_m \bar{W}_{jkh}^i - \frac{1}{6} [(\mathcal{B}_m g_{jh}) R_k^i + g_{jh} (\mathcal{B}_m R_k^i) \\ &+ (\mathcal{B}_m g_{jk}) R_h^i + g_{jk} (\mathcal{B}_m R_h^i)] \\ &+ \frac{1}{6} \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}). \end{aligned} \quad (3.2)$$

Using (2.6), in the equation (3.2) can be written as

$$\begin{aligned} \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_m \bar{W}_{jkh}^i + \frac{1}{6} \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &- \frac{1}{6} (g_{jh} (\mathcal{B}_m R_k^i) + g_{jk} (\mathcal{B}_m R_h^i)) \\ &+ \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jhm} + R_h^i C_{jkm}). \end{aligned} \quad (3.3)$$

By substituting equations (2.13) and (3.1) in to (3.3), we obtain:

$$\begin{aligned} \mathcal{B}_m \bar{W}_{jkh}^i &+ \frac{1}{6} \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) - \frac{1}{6} (g_{jh} (\mathcal{B}_m R_k^i) + \\ &g_{jk} (\mathcal{B}_m R_h^i)) + \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jhm} + R_h^i C_{jkm}) \\ &= \lambda_m \bar{W}_{jkh}^i + \frac{1}{6} \lambda_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &- \frac{1}{6} \lambda_m (g_{jh} R_k^i + g_{jk} R_h^i) + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \\ &\frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}). \end{aligned} \quad (3.4)$$

Alternatively, this can be expressed as:

$$\begin{aligned} \mathcal{B}_m \bar{W}_{jkh}^i &= \lambda_m \bar{W}_{jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk}) + \\ &\frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}) - \frac{1}{6} \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &+ \frac{1}{6} (g_{jh} (\mathcal{B}_m R_k^i) + g_{jk} (\mathcal{B}_m R_h^i)) \\ &- \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jhm} + R_h^i C_{jkm}) \\ &+ \frac{1}{6} \lambda_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) - \frac{1}{6} \lambda_m (g_{jh} R_k^i + g_{jk} R_h^i). \end{aligned} \quad (3.5)$$

This demonstrates that

$$\mathcal{B}_m \bar{W}_{jkh}^i = \lambda_m \bar{W}_{jkh}^i + \mu_m (\delta_k^i g_{jh} - \delta_h^i g_{jk})$$

$$\begin{aligned}
 & + \frac{1}{4} \gamma_m (W_k^i g_{jh} - W_h^i g_{jk}) \\
 & - \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jhm} + R_h^i C_{jkm}). \quad (3.6)
 \end{aligned}$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) & = \lambda_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}), \\
 \mathcal{B}_m R_k^i & = \lambda_m R_k^i, \text{ where } g_{jh} \neq 0, \text{ and} \\
 \mathcal{B}_m R_h^i & = \lambda_m R_h^i, \text{ where } g_{jk} \neq 0. \quad (3.7)
 \end{aligned}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 3.1. In the space $G^{2nd} \mathcal{B} \bar{W} - RF_n$, the M-projective curvature tensor \bar{W}_{jkh}^i represents a generalized recurrent Finsler space, provided that the condition (3.7) is satisfied.

By transecting equation (3.5) with y^j and utilizing equations (2.11a), (2.5b), (2.1a), (2.3) and (2.12a), we obtain the following result:

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_{kh}^i & = \lambda_m \bar{W}_{kh}^i + \mu_m (\delta_h^i y_k - \delta_k^i y_h) \\
 & + \frac{1}{4} \gamma_m (W_k^i y_h - W_h^i y_k) - \frac{1}{6} \mathcal{B}_m (\delta_k^i H_h + \delta_h^i H_k) \\
 & + \frac{1}{6} (y_h (\mathcal{B}_m R_k^i) + y_k (\mathcal{B}_m R_h^i)) \\
 & + \frac{1}{6} \lambda_m (\delta_k^i H_h + \delta_h^i H_k) - \frac{1}{6} \lambda_m (y_h R_k^i + y_k R_h^i). \quad (3.8)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_{kh}^i & = \lambda_m \bar{W}_{kh}^i + \mu_m (\delta_k^i y_h - \delta_h^i y_k) \\
 & + \frac{1}{4} \gamma_m (W_k^i y_h - W_h^i y_k). \quad (3.9)
 \end{aligned}$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m (\delta_k^i H_h + \delta_h^i H_k) & = \lambda_m (\delta_k^i H_h + \delta_h^i H_k), \\
 \mathcal{B}_m R_k^i & = \lambda_m R_k^i, \text{ where } y_h \neq 0, \text{ and} \\
 \mathcal{B}_m R_h^i & = \lambda_m R_h^i, \text{ where } y_k \neq 0. \quad (3.10)
 \end{aligned}$$

Therefore, the proof of theorem is completed, we conclude **Theorem 3.2.** In the space $G^{2nd} \mathcal{B} \bar{W} - RF_n$, the M-projective torsion tensor \bar{W}_{kh}^i (M-projective curvature tensor \bar{W}_{jkh}^i) represents a generalized recurrent Finsler space, provided that the condition (3.10) is satisfied.

By transvecting (3.8) with y^k and applying $(n = 4)$, along with equations (2.11b), (2.5a), (2.5b), (2.1b), (2.10a), (2.1c) and (2.12e), we obtain the following result:

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_h^i & = \lambda_m \bar{W}_h^i + \mu_m (y^i y_h - \delta_h^i F^2) \\
 & - \frac{1}{4} \gamma_m [W_h^i F^2] + \frac{1}{6} (y_h (\mathcal{B}_m R_k^i) y^k + F^2 (\mathcal{B}_m R_h^i)) \\
 & - \frac{1}{6} \mathcal{B}_m (y^i H_h + 3 \delta_h^i H) - \frac{1}{6} \lambda_m (y_h R_k^i y^k + F^2 R_h^i) \\
 & + \frac{1}{6} \lambda_m (y^i H_h + 3 \delta_h^i H). \quad (3.11)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_h^i & = \lambda_m \bar{W}_h^i + \mu_m (y^i y_h - \delta_h^i F^2) \\
 & - \frac{1}{4} \gamma_m [W_h^i F^2]. \quad (3.12)
 \end{aligned}$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m (y^i H_h + 3 \delta_h^i H) & = \lambda_m (y^i H_h + 3 \delta_h^i H), \\
 \mathcal{B}_m R_k^i & = \lambda_m R_k^i, \text{ where } y_h y^k \neq 0, \text{ and} \\
 \mathcal{B}_m R_h^i & = \lambda_m R_h^i, \text{ where } F^2 \neq 0. \quad (3.13)
 \end{aligned}$$

Thus, the proof of theorem is completed, we conclude **Theorem 3.3.** In the space $G^{2nd} \mathcal{B} \bar{W} - RF_n$, the M-projective deviation tensor \bar{W}_h^i represents a generalized recurrent Finsler

space if the tensors $(y^i H_h + 3 \delta_h^i H)$ and $(y_h R_k^i y^k + F^2 R_h^i)$ are generalized recurrent Finsler spaces.

By contracting the indices i and h in the equations (3.5), (3.8) and (3.11), and utilizing equations $(n = 4)$, (2.2a), (2.1a), (2.1b), (2.10b), (2.10c), (2.10d), (2.12c), (2.12d), (2.12e) and (2.1c), in conjunction with (2.11c), (2.11d) and (2.11e), we obtain the following result:

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_{jk} & = \lambda_m \bar{W}_{jk} + \mu_m (1 - n) g_{jk} + \frac{1}{4} \gamma_m [W_{jk}] \\
 & + \frac{1}{6} ((\mathcal{B}_m R_{jk}) + g_{jk} (\mathcal{B}_m R)) - \frac{1}{6} \mathcal{B}_m (1 + n) R_{jk} \\
 & + \frac{1}{6} \lambda_m (1 + n) R_{jk} - \frac{1}{6} \lambda_m (R_{jk} + g_{jk} R) \\
 & - \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jim} + R C_{jkm}). \quad (3.14)
 \end{aligned}$$

This demonstrates that

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_{jk} & = \lambda_m \bar{W}_{jk} + \mu_m (1 - n) g_{jk} + \frac{1}{4} \gamma_m [W_{jk}] \\
 & - \frac{1}{3} \mathcal{B}_q y^q (R_k^i C_{jim} + R C_{jkm}). \quad (3.15)
 \end{aligned}$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m R_{jk} & = \lambda_m R_{jk}, \text{ and} \\
 \mathcal{B}_m R & = \lambda_m R, \text{ where } g_{jk} \neq 0. \quad (3.16)
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{B}_m \bar{W}_k & = \lambda_m \bar{W}_k + \mu_m (1 - n) y_k \\
 & + \frac{1}{6} (y_i (\mathcal{B}_m R_k^i) + y_k (\mathcal{B}_m R)) - \frac{1}{6} \mathcal{B}_m (1 + n) H_k \\
 & - \frac{1}{6} \lambda_m (y_i R_k^i + y_k R) + \frac{1}{6} \lambda_m (1 + n) H_k. \quad (3.17)
 \end{aligned}$$

This demonstrates that

$$\mathcal{B}_m \bar{W}_k = \lambda_m \bar{W}_k + \mu_m (1 - n) y_k. \quad (3.18)$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m R_k^i & = \lambda_m R_k^i, \text{ where } y_i \neq 0, \\
 \mathcal{B}_m R & = \lambda_m R, \text{ where } y_k \neq 0, \text{ and} \\
 \mathcal{B}_m H_k & = \lambda_m H_k. \quad (3.19)
 \end{aligned}$$

In the last

$$\begin{aligned}
 \mathcal{B}_m \bar{W} & = \lambda_m \bar{W} + \mu_m (1 - n) F^2 \\
 & + \frac{1}{6} (y_i y^k (\mathcal{B}_m R_k^i) + F^2 (\mathcal{B}_m R)) - \frac{1}{2} (1 + n) \mathcal{B}_m H \\
 & - \frac{1}{6} \lambda_m (y_i R_k^i y^k + F^2 R) + \frac{1}{2} \lambda_m (1 + n) H. \quad (3.20)
 \end{aligned}$$

This demonstrates that

$$\mathcal{B}_m \bar{W} = \lambda_m \bar{W} + \mu_m (1 - n) F^2. \quad (3.21)$$

If and only if

$$\begin{aligned}
 \mathcal{B}_m R_k^i & = \lambda_m R_k^i, \text{ where } y_i y^k \neq 0, \\
 \mathcal{B}_m R & = \lambda_m R, \text{ where } F^2 \neq 0, \text{ and} \\
 \mathcal{B}_m H & = \lambda_m H. \quad (3.22)
 \end{aligned}$$

In conclusion the proof of theorem is completed, we can say **Theorem 3.4.** In the space $G^{2nd} \mathcal{B} \bar{W} - RF_n$, \bar{W} - Ricci tensor \bar{W}_{jk} , vector \bar{W}_k and scalar \bar{W} are defined in equations (3.15), (3.18) and (3.21), respectively, if and only if the conditions in equations (3.16), (3.19) and (3.22) are satisfied.

We introduce a new class of Finsler spaces, namely, generalized birecurrent spaces. These spaces generalize the concept of birecurrence to a broader setting and exhibit geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems.

Thus, this paper focuses on what is called the covariant derivative of second order.

By taking the covariant derivative of equation (3.3), with respect to in the sense of Berwald, the following results are obtained:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i + \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &- \frac{1}{6} \mathcal{B}_l (g_{jh} (\mathcal{B}_m R_k^i) + g_{jk} (\mathcal{B}_m R_h^i)) \\ &+ \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}). \end{aligned} \quad (3.23)$$

Or

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i - \frac{1}{6} ((\mathcal{B}_l g_{jh}) (\mathcal{B}_m R_k^i) + \\ &g_{jh} (\mathcal{B}_l \mathcal{B}_m R_k^i) + (\mathcal{B}_l g_{jk}) (\mathcal{B}_m R_h^i) + g_{jk} (\mathcal{B}_l \mathcal{B}_m R_h^i)) \\ &+ \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &+ \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}). \end{aligned} \quad (3.24)$$

The equation (3.24) can be expressed as

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i - \frac{1}{6} (g_{jh} (\mathcal{B}_l \mathcal{B}_m R_k^i) + \\ &g_{jk} (\mathcal{B}_l \mathcal{B}_m R_h^i)) + \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) - \\ &\frac{1}{6} ((\mathcal{B}_l g_{jh}) (\mathcal{B}_m R_k^i) + (\mathcal{B}_l g_{jk}) (\mathcal{B}_m R_h^i)) + \\ &\frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}). \end{aligned} \quad (3.25)$$

Using equation (2.6), equation (3.25) can be rewritten as follows:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m W_{jkh}^i &= \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i - \frac{1}{6} (g_{jh} (\mathcal{B}_l \mathcal{B}_m R_k^i) + \\ &g_{jk} (\mathcal{B}_l \mathcal{B}_m R_h^i)) + \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &+ \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}) \\ &+ \frac{1}{3} \mathcal{B}_q y^q (C_{jhl} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R_h^i)). \end{aligned} \quad (3.26)$$

Similarly, by applying equations (2.16) and (3.1) in (3.26), we obtain the result:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i &+ \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &- \frac{1}{6} (g_{jh} (\mathcal{B}_l \mathcal{B}_m R_k^i) + g_{jk} (\mathcal{B}_l \mathcal{B}_m R_h^i)) \\ &+ \frac{1}{3} \mathcal{B}_q y^q (C_{jhl} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R_h^i)) \\ &+ \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}) = a_{ml} \bar{W}_{jkh}^i \\ &+ \frac{1}{6} a_{ml} (\delta_k^i R_{jh} + \delta_h^i R_{jk}) + b_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &- \frac{1}{6} a_{ml} (g_{jh} R_k^i + g_{jk} R_h^i) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}) \\ &+ \frac{1}{4} c_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m (g_{jh} (\mathcal{B}_l W_k^i) - g_{jk} (\mathcal{B}_l W_h^i)) \\ &- 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}). \end{aligned} \quad (3.27)$$

Alternatively, the equation (3.27) can be expressed as:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i &= a_{ml} \bar{W}_{jkh}^i + b_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} c_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{4} \gamma_m (g_{jh} (\mathcal{B}_l W_k^i) - g_{jk} (\mathcal{B}_l W_h^i)) \\ &- 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}) \\ &- \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}) \\ &+ \frac{1}{6} a_{ml} (\delta_k^i R_{jh} + \delta_h^i R_{jk}) - \frac{1}{6} a_{ml} (g_{jh} R_k^i + g_{jk} R_h^i) \\ &- \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) \\ &+ \frac{1}{6} (g_{jh} (\mathcal{B}_l \mathcal{B}_m R_k^i) + g_{jk} (\mathcal{B}_l \mathcal{B}_m R_h^i)) \\ &- \frac{1}{3} \mathcal{B}_q y^q (C_{jhl} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R_h^i)) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}). \end{aligned} \quad (3.28)$$

This demonstrates that

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i &= a_{ml} \bar{W}_{jkh}^i + b_{ml} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \\ &+ \frac{1}{4} c_{ml} (W_k^i g_{jh} - W_h^i g_{jk}) \\ &+ \frac{1}{4} \gamma_m (g_{jh} (\mathcal{B}_l W_k^i) - g_{jk} (\mathcal{B}_l W_h^i)) \\ &- 2\mu_m \mathcal{B}_q y^q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}) \\ &- 2\mu_m y^q \mathcal{B}_q (\delta_k^i C_{jhl} - \delta_h^i C_{jkl}) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_k^i C_{jhl} - W_h^i C_{jkl}) \\ &- \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jhm} + R_h^i C_{jkm}) \\ &- \frac{1}{3} \mathcal{B}_q y^q (C_{jhl} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R_h^i)). \end{aligned} \quad (3.29)$$

If and only if

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m (\delta_k^i R_{jh} + \delta_h^i R_{jk}) &= a_{ml} (\delta_k^i R_{jh} + \delta_h^i R_{jk}), \\ \mathcal{B}_l \mathcal{B}_m R_k^i &= a_{ml} R_k^i, \text{ where } g_{jh} \neq 0, \text{ and} \\ \mathcal{B}_l \mathcal{B}_m R_h^i &= a_{ml} R_h^i, \text{ where } g_{jk} \neq 0. \end{aligned} \quad (3.30)$$

In conclusion the proof of theorem is completed, we can determine

Theorem 3.5. In the space $G^{2nd} \mathcal{B} \bar{W} - \text{BRF}_n$, the M-projective curvature tensor \bar{W}_{jkh}^i defines a generalized birecurrent Finsler space if and only if the condition in equation (3.30) is satisfied.

By transvecting equation (3.28) with y^j and utilizing equations (2.11a), (2.5b), (2.1a), (2.3) and (2.12a), we obtain the following result:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{kh}^i &= a_{ml} \bar{W}_{kh}^i + b_{ml} (\delta_k^i y_h - \delta_h^i y_k) \\ &+ \frac{1}{4} c_{ml} (W_k^i y_h - W_h^i y_k) \\ &- \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (\delta_k^i H_h + \delta_h^i H_k) + \frac{1}{6} (y_h (\mathcal{B}_l \mathcal{B}_m R_k^i) + \\ &y_k (\mathcal{B}_l \mathcal{B}_m R_h^i)) + \frac{1}{6} a_{ml} (\delta_k^i H_h + \delta_h^i H_k) \\ &- \frac{1}{6} a_{ml} (y_h R_k^i + y_k R_h^i) + \frac{1}{4} \gamma_m (y_h (\mathcal{B}_l W_k^i) - \\ &y_k (\mathcal{B}_l W_h^i)). \end{aligned} \quad (3.31)$$

This demonstrates that

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{kh}^i &= a_{ml} \bar{W}_{kh}^i + b_{ml} (\delta_k^i y_h - \delta_h^i y_k) \\ &+ \frac{1}{4} c_{ml} (W_k^i y_h - W_h^i y_k) \\ &+ \frac{1}{4} \gamma_m (y_h (\mathcal{B}_l W_k^i) - y_k (\mathcal{B}_l W_h^i)). \end{aligned} \quad (3.32)$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m (\delta_k^i H_h + \delta_h^i H_k) = a_{ml} (\delta_k^i H_h + \delta_h^i H_k),$$

$$\mathcal{B}_l \mathcal{B}_m R_k^i = a_{ml} R_k^i, \text{ where } y_h \neq 0, \text{ and}$$

$$\mathcal{B}_l \mathcal{B}_m R_h^i = a_{ml} R_h^i, \text{ where } y_k \neq 0. \quad (3.33)$$

Therefore, the proof of theorem is completed, we conclude Theorem 3.6. In the space $G^{2nd} \mathcal{B} \bar{W} - BRF_n$, the covariant derivative of the second orders for the M-projective torsion tensor \bar{W}_{kh}^i (M-projective curvature tensor \bar{W}_{jkh}^i) defines a generalized birecurrent Finsler space if and only if the condition in equation (3.33) is satisfied.

By transvecting condition (3.31) with y^k and applying ($n = 4$), along with equations (2.11b), (2.5a), (2.5b), (2.1b), (2.10a), (2.1c) and (2.12e), we obtain the following result:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_h^i &= a_{ml} \bar{W}_h^i + b_{ml} (y^i y_h - \delta_h^i F^2) \\ &- \frac{1}{4} c_{ml} (W_h^i F^2) - \frac{1}{4} \gamma_m F^2 (\mathcal{B}_l W_h^i) \\ &- \frac{1}{6} (y_h y^k (\mathcal{B}_l \mathcal{B}_m R_k^i) + F^2 (\mathcal{B}_l \mathcal{B}_m R_h^i)) \\ &+ \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (y^i H_h + 3\delta_h^i H) \\ &+ \frac{1}{6} a_{ml} (y_h R_k^i y^k + F^2 R_h^i) \\ &- \frac{1}{6} a_{ml} (y^i H_h + 3\delta_h^i H). \end{aligned} \quad (3.34)$$

This demonstrates that

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_h^i &= a_{ml} \bar{W}_h^i + b_{ml} (y^i y_h - \delta_h^i F^2) \\ &- \frac{1}{4} c_{ml} (F^2 W_h^i) - \frac{1}{4} \gamma_m F^2 (\mathcal{B}_l W_h^i). \end{aligned} \quad (3.35)$$

If and only if

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m (y^i H_h + 3\delta_h^i H) &= a_{ml} (y^i H_h + 3\delta_h^i H), \\ \mathcal{B}_l \mathcal{B}_m R_k^i &= a_{ml} R_k^i, \text{ where } y_h y^k \neq 0, \text{ and} \\ \mathcal{B}_l \mathcal{B}_m R_h^i &= a_{ml} R_h^i, \text{ where } F^2 \neq 0. \end{aligned} \quad (3.36)$$

Therefore, the proof of theorem is completed, we can say Theorem 3.7. In the space $G^{2nd} \mathcal{B} \bar{W} - BRF_n$, the M-projective deviation tensor \bar{W}_h^i represents a generalized birecurrent Finsler space if the tensors $(y^i H_h + 3\delta_h^i H)$ and $(y_h R_k^i y^k + F^2 R_h^i)$ are generalized birecurrent Finsler space.

By contracting the indices i and h in the equations (3.28), (3.31) and (3.34), and utilizing equations ($n = 4$), (2.2a), (2.1a), (2.1b), (2.10b), (2.10c), (2.10d), (2.12c), (2.12d),

(2.12e) and (2.1c), along with the relations in equations (2.11c), (2.11d) and (2.11e), we obtain the following result:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{jk} &= a_{ml} \bar{W}_{jk} + b_{ml} (1 - n) g_{jk} + \frac{1}{4} c_{ml} W_{jk} \\ &- \frac{1}{6} (1 + n) \mathcal{B}_l \mathcal{B}_m R_{jk} + \frac{1}{6} ((\mathcal{B}_l \mathcal{B}_m R_{jk}) + g_{jk} (\mathcal{B}_l \mathcal{B}_m R)) \\ &- \frac{1}{3} \mathcal{B}_q y^q (C_{jil} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R)) \\ &+ \frac{1}{6} a_{ml} (1 + n) R_{jk} - \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jim} + R C_{jkm}) \\ &+ \frac{1}{4} \gamma_m \mathcal{B}_l W_{jk} - \frac{1}{2} \gamma_m \mathcal{B}_q y^q W_k^i C_{jil} \\ &- 2\mu_m y^q \mathcal{B}_q (1 - n) C_{jkl} - \frac{1}{6} a_{ml} (R_{jk} + g_{jk} R). \end{aligned} \quad (3.37)$$

This demonstrates that

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{jk} &= a_{ml} \bar{W}_{jk} + b_{ml} (1 - n) g_{jk} + \frac{1}{4} c_{ml} W_{jk} \\ &+ \frac{1}{4} \gamma_m \mathcal{B}_l W_{jk} - \frac{1}{2} \gamma_m \mathcal{B}_q y^q W_k^i C_{jil} \end{aligned}$$

$$\begin{aligned} &- \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_k^i C_{jim} + R C_{jkm}) \\ &- \frac{1}{3} \mathcal{B}_q y^q (C_{jil} (\mathcal{B}_m R_k^i) + C_{jkl} (\mathcal{B}_m R)) \\ &- 2\mu_m y^q \mathcal{B}_q (1 - n) C_{jkl}. \end{aligned} \quad (3.38)$$

If and only if

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m R_{jk} &= a_{ml} R_{jk}, \text{ and} \\ \mathcal{B}_l \mathcal{B}_m R &= a_{ml} R, \text{ where } g_{jk} \neq 0. \end{aligned} \quad (3.39)$$

And $\mathcal{B}_l \mathcal{B}_m \bar{W}_k = a_{ml} \bar{W}_k + b_{ml} (1 - n) y_k$

$$\begin{aligned} &+ \frac{1}{6} (y_i (\mathcal{B}_l \mathcal{B}_m R_k^i) + y_k (\mathcal{B}_l \mathcal{B}_m R)) - \frac{1}{6} (1 + n) \mathcal{B}_l \mathcal{B}_m H_k \\ &- \frac{1}{6} a_{ml} (y_i R_k^i + y_k R) + \frac{1}{6} a_{ml} (1 + n) H_k. \end{aligned} \quad (3.40)$$

This demonstrates that

$$\mathcal{B}_l \mathcal{B}_m \bar{W}_k = a_{ml} \bar{W}_k + b_{ml} (1 - n) y_k. \quad (3.41)$$

If and only if

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m H_k &= a_{ml} H_k, \\ \mathcal{B}_l \mathcal{B}_m R_k^i &= a_{ml} R_k^i, \text{ where } y_i \neq 0, \text{ and} \\ \mathcal{B}_l \mathcal{B}_m R &= a_{ml} R, \text{ where } y_k \neq 0. \end{aligned} \quad (3.42)$$

In the last

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W} &= a_{ml} \bar{W} + b_{ml} (1 - n) F^2 \\ &- \frac{1}{6} (y_i y^k (\mathcal{B}_l \mathcal{B}_m R_k^i) + F^2 (\mathcal{B}_l \mathcal{B}_m R)) + \frac{1}{2} (n + 1) \mathcal{B}_l \mathcal{B}_m H \\ &+ \frac{1}{6} a_{ml} (y_i R_k^i y^k + F^2 R) - \frac{1}{2} a_{ml} (n + 1) H. \end{aligned} \quad (3.43)$$

This demonstrates that

$$\mathcal{B}_l \mathcal{B}_m \bar{W} = a_{ml} \bar{W} + b_{ml} (n - 1) F^2. \quad (3.44)$$

If and only if

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m H &= a_{ml} H, \\ \mathcal{B}_l \mathcal{B}_m R_k^i &= a_{ml} R_k^i, \text{ where } y_i y^k \neq 0, \text{ and} \\ \mathcal{B}_l \mathcal{B}_m R &= a_{ml} R, \text{ where } F^2 \neq 0. \end{aligned} \quad (3.45)$$

In conclusion the proof of theorem is completed, we can say Theorem 3.8. In the space $G^{2nd} \mathcal{B} \bar{W} - BRF_n$, the M-projective Ricci tensor \bar{W}_{jk} , vector \bar{W}_k and scalar \bar{W} are defined in equations (3.38), (3.41) and (3.44), respectively, provided that the conditions in equations (3.39), (3.42) and (3.45) are satisfied.

By transvecting equation (3.28) with g_{ir} and utilizing equations (2.2b), (2.10d), (2.12d) and (2.11f), we obtain the following result:

$$\begin{aligned} \mathcal{B}_l \mathcal{B}_m \bar{W}_{rjkh} &= a_{ml} \bar{W}_{rjkh} + b_{ml} (g_{rk} g_{jh} - g_{rh} g_{jk}) \\ &- \frac{1}{6} \mathcal{B}_l \mathcal{B}_m (g_{rk} R_{jh} + g_{rh} R_{jk}) \\ &+ \frac{1}{6} (g_{jh} (\mathcal{B}_l \mathcal{B}_m R_{rk}) + g_{jk} (\mathcal{B}_l \mathcal{B}_m R_{rh})) \\ &- \frac{1}{3} y^q \mathcal{B}_q (C_{jhl} (\mathcal{B}_m R_{rk}) + C_{jkl} (\mathcal{B}_m R_{rh})) \\ &- \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_{rk} C_{jhm} + R_{rh} C_{jkm}) \\ &+ \frac{1}{6} a_{ml} (g_{rk} R_{jh} + g_{rh} R_{jk}) - \frac{1}{6} a_{ml} (g_{jh} R_{rk} + g_{jk} R_{rh}) \\ &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_{rk}) g_{jh} - (\mathcal{B}_l W_{rh}) g_{jk}) \\ &+ \frac{1}{4} c_{ml} (W_{rk} g_{jh} - W_{rh} g_{jk}) \\ &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_{rk} C_{jhl} - W_{rh} C_{jkl}) \\ &- 2\mu_m y^q \mathcal{B}_q (g_{rk} C_{jhl} - g_{rh} C_{jkl}). \end{aligned} \quad (3.46)$$

This demonstrates that

$$\begin{aligned}
 \mathcal{B}_l \mathcal{B}_m \bar{W}_{rjkh} &= a_{ml} \bar{W}_{rjkh} + b_{ml} (g_{rk} g_{jh} - g_{rh} g_{jk}) \\
 &+ \frac{1}{4} \gamma_m ((\mathcal{B}_l W_{rk}) g_{jh} - (\mathcal{B}_l W_{rh}) g_{jk}) \\
 &+ \frac{1}{4} c_{ml} (W_{rk} g_{jh} - W_{rh} g_{jk}) \\
 &- \frac{1}{2} \gamma_m \mathcal{B}_q y^q (W_{rk} C_{jhl} - W_{rh} C_{jkl}) \\
 &- 2\mu_m y^q \mathcal{B}_q (g_{rk} C_{jhl} - g_{rh} C_{jkl}) \\
 &- \frac{1}{3} y^q \mathcal{B}_q (C_{jhl} (\mathcal{B}_m R_{rk}) + C_{jkl} (\mathcal{B}_m R_{rh})) \\
 &- \frac{1}{3} y^q \mathcal{B}_q \mathcal{B}_l (R_{rk} C_{jhm} + R_{rh} C_{jkm}). \tag{3.47}
 \end{aligned}$$

If and only if

$$\begin{aligned}
 \mathcal{B}_l \mathcal{B}_m (g_{rk} R_{jh} + g_{rh} R_{jk}) &= a_{ml} (g_{rk} R_{jh} + g_{rh} R_{jk}), \\
 \mathcal{B}_l \mathcal{B}_m R_{rk} &= a_{ml} R_{rk}, \text{ where } g_{jh} \neq 0, \text{ and} \\
 \mathcal{B}_l \mathcal{B}_m R_{rh} &= a_{ml} R_{rh}, \text{ where } g_{jk} \neq 0. \tag{3.48}
 \end{aligned}$$

Thus, the proof of theorem is completed, we conclude

Theorem 3.9. In the space $G^{2nd} \mathcal{B}\bar{W} - BRF_n$, M-projective associate tensor \bar{W}_{rjkh} (M-projective curvature tensor \bar{W}_{jkh}^i) characterizes a generalized birecurrent Finsler space, provided that the condition (3.48) is satisfied.

CONCLUSION

This paper introduces and explores several new properties of the M-projective curvature tensor, \bar{W}_{jkh}^i , in the context of Finsler geometry, specifically within the framework of generalized recurrent Finsler spaces. By deriving a series of equations and theorems, we have demonstrated the underlying relationships between various curvature tensors and their conditions for being generalized recurrent Finsler spaces.

The key findings of the study are as follows:

- Generalized Recurrent Finsler Spaces:** We have established that in the space $G^{2nd} \mathcal{B}\bar{W} - BRF_n$, the M-projective curvature tensor \bar{W}_{jkh}^i represents a generalized recurrent Finsler space, provided that certain conditions, such as equation (3.7), are satisfied. This is crucial as it extends the concept of recurrent Finsler spaces and provides a deeper understanding of their geometric properties.
- M-projective Torsion Tensor:** Through detailed derivations and the application of transvection techniques, we showed that the M-projective torsion tensor \bar{W}_{jkh}^i in a similar manner also represents a generalized recurrent Finsler space, subject to specific conditions (3.10). This adds a new layer to the study of torsion tensors in Finsler geometry, highlighting their potential to model more complex geometric structures.
- Deviations in Geometrical Properties:** The study also presents a comprehensive approach to the M-projective deviation tensor \bar{W}_h^i . By incorporating deviations into the curvature analysis, we have established that the deviation tensor represents a generalized recurrent Finsler space under conditions (3.13), thus providing a more generalized framework for the classification of Finsler spaces.
- Curvature Tensors and Their Interactions:** By examining various covariant derivatives and transvecting equations, we have also shown how the covariant derivatives $\mathcal{B}_m \bar{W}_{jkh}^i$ and

$\mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i$ contribute to understanding the interplay between different curvature tensors in the context of generalized recurrent Finsler spaces. The results underscore the importance of these interactions in establishing a more robust classification of Finsler spaces and contribute to the understanding of their broader geometric properties.

5. Implications for Birecurrent Finsler Spaces: The concept of generalized birecurrent Finsler spaces is introduced, providing an extension to the classical idea of birecurrence. This new class of spaces offers an enriched set of geometrical structures with applications in higher-order differential geometry and theoretical physics, particularly in the study of spaces with more intricate curvatures and torsions.

6. New Theorems and Classifications: The paper provides the proof and theorems associated with the space $G^{2nd} \mathcal{B}\bar{W} - BRF_n$ and the conditions under which the M-projective curvature tensor and its derivatives define generalized recurrent or birecurrent Finsler spaces. These results extend the existing framework of Finsler geometry by providing more generalized conditions for classification.

In conclusion, the results of this study open several new avenues for future research, particularly in understanding more complex and higher-dimensional Finsler spaces and their applications. The proposed definitions and conditions lay the groundwork for further investigation into generalized recurrent Finsler spaces, M-projective tensors, and their relations to curvature, torsion, and deviation. Moreover, the introduction of generalized birecurrent spaces offers a promising direction for the exploration of spaces with highly intricate geometric structures.

RECOMMENDATIONS

Based on the comprehensive derivations and proofs established within this study, the following recommendations are proposed for future research directions in the study of M-projective curvature tensors and generalized Finsler spaces:

- Further Analysis of Generalized Finsler Spaces:**

The results obtained for generalized recurrent Finsler spaces, specifically in relation to the M-projective curvature tensor \bar{W}_{jkh}^i , offer potential for further investigation. Future studies could explore additional conditions under which these spaces exhibit notable geometric properties such as curvature, torsion, and deviation. It would be beneficial to extend the characterization of these spaces to higher-dimensional settings (i.e., beyond $(n = 4)$ and analyze their applications in complex geometries.

- Refinement of Covariant Derivatives in Second-Order Calculations:**

In this work, we have introduced covariant derivatives such as $\mathcal{B}_l \mathcal{B}_m \bar{W}_{jkh}^i$ and related expressions. These derivatives offer a systematic approach to studying the geometry of M-projective tensors. However, more advanced techniques could be developed to refine the computational process, particularly for higher-order covariant derivatives. Exploring the use of alternative differential operators might yield more efficient methods for deriving related geometric invariants.

- Potential Applications to Generalized Birecurrent Spaces:**

The concept of generalized birecurrent spaces is introduced as a potential extension of classical birecurrent

Finsler spaces. This new class of spaces requires deeper exploration of its geometric and physical interpretations, especially in contexts where Finsler spaces with curvature constraints are applicable. We suggest investigating the applicability of these spaces in areas such as theoretical physics, where the structure of spacetime may be modeled using advanced differential geometry.

4. Numerical Simulations and Computational Methods:

While the theoretical framework presented in this paper provides a solid foundation, numerical simulations could offer insights into the practical implications of the results. Computational methods for solving systems of differential equations governing the M-projective curvature tensors may lead to new insights into the behavior of generalized Finsler spaces under various boundary conditions. This would further enhance the understanding of the relationship between curvature tensors and geometric properties in high-dimensional spaces.

5. Integration of Torsion and Curvature in Physical Models:

Torsion and curvature play a significant role in the study of differential geometry, particularly in the context of general relativity and other physical theories. Future research could focus on how these tensors interact in the context of Finsler spaces with torsion, especially when considering specific models of spacetime or material media. The results of this study, especially the properties of \bar{W}_{jkh}^i and its transvecting forms, could serve as the foundation for developing new models of physical phenomena.

6. Extension to Non-Riemannian Geometries:

The framework of M-projective curvature tensors could also be extended to non-Riemannian geometries, such as those encountered in quantum gravity and string theory. By relaxing some of the assumptions inherent to classical Riemannian geometry, future studies could explore how curvature tensors behave in more generalized spaces and whether this yields new insights into the fundamental nature of space and time.

These recommendations aim to guide future investigations into the properties and applications of generalized Finsler spaces and M-projective curvature tensors, contributing to both theoretical advancements and practical implementations in geometric and physical models.

REFERENCES

- [1] H. Abu-Donia, S. Shenawy, and A. Abdehameed, "The w^* -curvature tensor on relativistic space-times," *Kyungpook Math. J.*, vol. 60, pp. 185–195, 2020.
- [2] Z. Ahsan and M. Ali, "On some properties of w -curvature tensor," *Palestine J. Math.*, vol. 3, no. 1, pp. 61–69, 2014.
- [3] Z. Ahsan and M. Ali, "Curvature tensor for the spacetime of general relativity," *Palestine J. Math.*, vol. 2, pp. 1–15, 2016.
- [4] A. M. Al-Qashbari, A. A. Abdallah, and S. M. Baleedi, "A study of M-projective curvature tensor \bar{W}_{jkh}^i in GBK-5RF_n via Lie derivative," *Int. J. Appl. Sci. Math. Theory*, vol. 11, no. 1, pp. 1–9, 2025.
- [5] A. M. Al-Qashbari, A. A. Abdallah, and F. Abdallah, "Decomposition of generalized recurrent tensor fields of R^h -nth order in Finsler manifolds," *J. Sci. Technol.*, vol. 30, no. 2, pp. 99–105, 2025.
- [6] A. M. A. AL-Qashbari, A. A. Abdallah, and K. S. Nasr, "Generalized of h-torsion and curvature structures in generalized recurrent space of third order by Cartan covariant derivatives," *GPH-Int. J. Math.*, vol. 8, no. 2, pp. 1–9, 2025.
- [7] A. M. Al-Qashbari, A. A. Abdallah, and F. A. Al-ssallal, "Recurrent Finsler structures with higher-order generalizations defined by special curvature tensors," *Int. J. Adv. Res. Sci. Commun. Technol.*, vol. 4, no. 1, pp. 68–75, 2024.
- [8] A. M. Al-Qashbari and F. A. Al-ssallal, "Study of curvature tensor by using Berwald's and Cartan's higher-order derivatives in Finsler space," *Technol. Appl. Humanit. Acad. J.*, vol. 1, no. 1, pp. 1–15, 2024.
- [9] A. M. Al-Qashbari, M. Haoues, and F. A. Al-ssallal, "A decomposition analysis of Weyl's curvature tensor via Berwald's first and second order derivatives in Finsler spaces," *J. Innov. Appl. Math. Comput. Sci.*, vol. 4, no. 2, pp. 201–213, 2024.
- [10] A. M. Al-Qashbari, A. A. Abdallah, and S. M. Baleedi, "Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space," *GPH-Int. J. Math.*, vol. 8, no. 1, pp. 24–32, 2025.
- [11] A. M. A. AL-Qashbari and F. Y. A. Qasem, "Study on generalized BR-trirecurrent Finsler space," *J. Yemen Eng.*, vol. 15, pp. 79–89, 2017.
- [12] A. M. A. AL-Qashbari, "On generalized for curvature tensors P_{jkh}^i of second order in Finsler space," *Univ. Aden J. Nat. Appl. Sci.*, vol. 24, no. 1, pp. 171–176, 2020.
- [13] A. M. A. AL-Qashbari, "Some properties for Weyl's projective curvature tensors of generalized W_{h} -birecurrent in Finsler spaces," *Univ. Aden J. Nat. Appl. Sci.*, vol. 23, no. 1, pp. 181–189, 2019.
- [14] A. M. A. AL-Qashbari, "Some identities for generalized curvature tensors in B-Recurrent Finsler space," *J. New Theory*, no. 32, pp. 30–39, 2020.
- [15] A. M. A. AL-Qashbari, "Recurrence decompositions in Finsler space," *J. Math. Anal. Model.*, vol. 1, pp. 77–86, 2020.
- [16] A. M. A. AL-Qashbari and A. A. S. AL-Maisary, "Study on generalized W_{jkh}^i of fourth order recurrent in Finsler space," *J. Yemeni Eng., Univ. Aden*, vol. 17, no. 2, pp. 72–86, 2023.
- [17] A. M. A. AL-Qashbari, A. A. Abdallah, and K. S. Nasr, "On generalized tri recurrent space by using Gh-covariant derivative in Finsler geometry," *J. Math. Probl. Equ. Stat.*, vol. 6, no. 1, pp. 91–100, 2025.
- [18] A. M. A. AL-Qashbari, S. Saleh, and I. Ibedou, "On some relations of R-projective curvature tensor in recurrent Finsler space," *J. Non-Linear Model. Anal. (China)*, vol. 6, no. 4, pp. 1216–1227, 2024.
- [19] S. M. S. Baleedi, "On certain generalized BK-recurrent Finsler space," M.S. thesis, Univ. Aden, Aden, Yemen, 2017.
- [20] B. Misra, S. B. Misra, K. Srivastava, and R. B. Srivastava, "Higher order recurrent Finsler spaces with

- Berwald's curvature tensor field," *J. Chem. Biol. Phys. Sci.*, vol. 4, no. 1, pp. 624–631, 2014.
- [21] A. Goswami, "A study of certain types of special Finsler spaces in differential geometry: Systematic review," *J. Math. Appl. Sci. Technol.*, vol. 9, no. 1, pp. 23–30, 2017.
- [22] P. N. Pandey, S. Saxena, and A. Goswami, "On a generalized H-recurrent space," *J. Int. Acad. Phys. Sci.*, vol. 15, pp. 201–211, 2011.
- [23] H. Rund, *The Differential Geometry of Finsler Spaces*, 2nd ed. Berlin: Springer-Verlag, 1981. [In Russian], Nauka, Moscow.