ON EXTENSIONS OF EXTENDED BETA AND GAUSS HYPERGEOMETRIC FUNCTIONS WITH THEIR APPLICATIONS

M. A. H Kulib(1)*
F. B. F. Mohsen(2)

Received: 05/03/2025
Revised: 23/03/2025
Accepted: 24/03/2025

---

[1] Department of Basic Sciences, Faculty of Engineering, Aden University, Yemen.
[2] Department of Mathematics, Faculty of Education, Abian University, Yemen.
*Corresponding Author's Email: Maisoonkulib@gmail.com

On Extensions of Extended Beta and Gauss Hypergeometric Functions with Their Applications

M. A. H. Kulib
Department of Basic Sciences,
Faculty of Engineering,
Aden University,
Yemen
Maisoonkulib@gmail.com

F. B. F. Mohsen
Department of Mathematics,
Faculty of Education,
Abian University,
Yemen

Abstract— The Beta function and its extensions play a crucial role in the study of some special functions; several researchers have introduced and investigated various extensions of this important function. In this paper we present further extensions of the Beta and the Gauss hypergeometric functions. We also explore some of their properties, derive integral representation, and establish summation formulas. Furthermore, we investigate the relationship between these generalized Beta functions and other special functions, such as the Fox Wright function, Fox's H-function, and generalized hypergeometric function.

Keywords— Beta function, Extended Beta function, Hypergeometric function, Extended Hypergeometric function, Mittag-Leffler function.

I. INTRODUCTION

We start with the classical Beta and Gamma functions defined respectively as follows (see [22], [27]):

$$B(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}\,du, \qquad (1.1)$$
$$(Re(x)>0, Re(y)>0\,),$$

and

$$\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}\,du, \qquad (Re(x)>0). \qquad (1.2)$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad (Re(x)>0, Re(y)>0). \quad (1.3)$$

The Gauss hypergeometric function and its integral representation defined respectively as follows (see [22, p. 85]):

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \qquad (1.4)$$
$$(|z|<1, \alpha,\beta,\gamma\in\mathbb{C}\ and\ \gamma\neq 0,-1,-2,-3,\dots),$$

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}$$
$$\times\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-\alpha}du, \qquad (1.5)$$
$$(Re(\gamma)>Re(\beta)>0, |arg(1-z)|<\pi).$$

The Mittag-Leffler function and its diverse generalizations are important, in particular, in connection with the theories of fractional calculus and special functions. Various researchers have presented and explored several extensions and generalization of Gamma and Beta functions related to the Mittag-Leffler function have been investigated (see [1-12], [14-8], [13-17], [18-21], [23-27]).

Shadab et al. [26] introduced a new extension of Beta function as:

$$B_p^{\alpha}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}E_\alpha\left(-\frac{p}{u(1-u)}\right)du, \quad (1.6)$$

Where $Re(x)>0, Re(y)>0$ and $E_\alpha(.)$ is the Mittag-Leffler function is defined as follows: (see [17])

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \qquad Re(\alpha)>0, \qquad z\in\mathbb{C}. \quad (1.7)$$

Obviously, for $\alpha=1, p=0$,(1.6) reduces to (1.1).

In the same paper, they defined the following integral representations of extended hypergeometric function as:

$$F_p^{\alpha}(\alpha,\beta;\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}$$
$$\times\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-\alpha}E_\alpha\left(\frac{-p}{u(1-u)}\right)du, \qquad (1.8)$$
$$(Re(p)>0, Re(\gamma)>Re(\beta)>0, |arg(1-z)|<\pi).$$

Obviously, for $\alpha=1, p=0$, (1.8) reduces to (1.5).

Goyal et al. [10] provide the following extension of Beta function by treating the Waiman function (two parameter's Mittag- Leffler function) as kernel:

$$B_{k_1,k_2}^{p}(x,y)$$
$$=\int_0^1 u^{x-1}(1-u)^{y-1}E_{k_1,k_2}\left(-\frac{p}{u(1-u)}\right)du, \qquad (1.9)$$

Where $Re(x)>0, Re(y)>0$, $k_1,k_2\in R_o^+$, and $E_{k_1,k_2}(.)$ is defined as (see [26])

$$E_{k_1,k_2}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(k_1 n+k_2)}. \qquad (1.10)$$

Al Gonah and Mohammed [2] used the generalized Mittag-Leffler function introduced by Prabhaker to present the following extended of Beta functions as:

$$B_p^{(\alpha,\beta,\gamma)}(x,y)$$
$$=\int_0^1 t^{x-1}(1-t)^{y-1}\,E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{u(1-u)}\right)du, \qquad (1.11)$$

Where $Re(x)>0, Re(y)>0\,, Re(\alpha)>0, Re(p)\geq 0,$ and $E_{\alpha,\beta}^{\gamma}(.)$ *is* Mittag-Leffler function which defined in Prabhakar [21] as:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \qquad (1.12)$$

Where $Re(\alpha)>0, \alpha, \beta, \gamma \in \mathbb{C}$.

Abubakar and Kabara [1] study and studied the following extended Beta function by using the four- parameter Mittag-Leffler function (Salim function):

$$B_{k_1,k_2,k_3}^{k_4,p}(x,y)$$
$$=\int_0^1 u^{x-1}(1-u)^{y-1}\, E_{k_1,k_2,k_3}^{k_4}\left(-\frac{p}{u(1-u)}\right)du, \quad (1.13)$$

Where $Re(x)>0, Re(y)>0\ , k_1, k_2, k_3, k_4 \in R_0^+, Re(p)>0$.

and $\mathrm{E}_{k_1,k_2,k_3}^{k_4}(.)$ is defined as follows:(see [24])

$$E_{k_1,k_2,k_3}^{k_4}(z)=\sum_{n=0}^{\infty}\frac{(k_3)_n}{(k_4)_n\Gamma(k_1 n+k_2)}z^n, \qquad (1.14)$$

Where,

$$k_1, k_2, k_3, k_4 \in \mathbb{C}, Re(k_1)>0, Re(k_2)>0, Re(k_3)>0, Re(k_4)>0.$$

Dudi and Abubakar [7] study another extension of Beta functions related to the generalized Mittag-Leffler function

$$B_{k_1,k_2,p,\omega}^{k_3,k_4,q}(x,y)$$
$$=\int_0^1 u^{x-1}(1-u)^{y-1}\, E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega}{u(1-u)}\right)du, \quad (1.15)$$

Where,$Re(x)>0, Re(y)>0\ , k_1, k_2, k_3, k_4 \in R_0^+, \omega \geq 0$,
$E_{k_1,k_2,p}^{k_3,k_4,q}(.)$ the generalized Mittag-Leffler function is introduced by Salim and Faraj, as follows: (see [25])

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z)=\sum_{n=0}^{\infty}\frac{(k_3)_{qn}}{(k_4)_{pn}\Gamma(k_1 n+k_2)}z^n, \qquad (1.16)$$

Where,

$$k_1, k_2, k_3, k_4 \in \mathbb{C}, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\}>0, p, q>0, q<Re(k_1)+p.$$

Throughout this paper, we need the following well-known facts and rules.

Wright generalized hypergeometric function (Fox- Wright function ${}_p\psi_q$)(see [27])

$${}_p\psi_q\left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_q); \\ (\beta_1, B_1), \dots, (\beta_q, B_q);\end{matrix}\, z\right]$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j n)z^n}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)n!}, \qquad (1.17)$$

where the coefficients $A_j \in \mathbb{R}^+(j=1, \dots, p)$ and $B_j \in \mathbb{R}^+(j=1, \dots, q)$such that

$$1+\sum_{n=0}^{q}B_j-\sum_{n=0}^{p}A_j \geq 0.$$

A special case of (1.17) reduce to the generalized hypergeometric function ${}_p F_q$

$${}_p\psi_q\left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_q); \\ (\beta_1, B_1), \dots, (\beta_q, B_q);\end{matrix}\, z\right]$$
$$=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\, {}_p F_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q;\end{matrix}\, z\right]. \qquad (1.18)$$

Fox- H- function $H_{p,q}^{m,n}$ (see [13])

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, \alpha_1), \dots, (a_p, \alpha_q) \\ (b_1, \beta_1), \dots, (b_q, \beta_q)\end{matrix}\right.\right]$$
$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{q}\Gamma(a_j-\alpha_j s)}\, z^{-s}ds \ (1.19)$$

2. FURTHER EXTENSION OF BETA FUNCTION AND ITS PROPERTIES

In this section, we define a new generalization of Beta function as follows:

Definition 2.1. The Further extension of extended Beta function is defined as:

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)$$
$$=\int_0^1 u^{x-1}(1-u)^{y-1}\, E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du, \quad (2.1)$$

Where$Re(x)>0, Re(y)>0, Re(\omega_1)>0, Re(\omega_2)>0$, $k_1, k_2, k_3, k_4 \in \mathbb{C}$, $min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\}>0, p, q>0, q<Re(k_1)+p$ and $E_{k_1,k_2,p}^{k_3,k_4,q}(.)$ is the generalized Mittag- Leffler function defined in (1.16).

Remark 2.1. Note that:

(i) If $\omega_1=\omega_2$ then (2.1) reduces to the well-known extended Beta function (1.15) given by Dudi and Abubakar [7].

3. PROPERTIES OF EXTENDED BETA FUNCTION

In this section, we investigate various properties of the extended Beta function $B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)$ as follows:

Theorem 3.1. The extension of Beta function satisfies the following functional relation

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x+1,y)+B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y+1)$$
$$=B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y) \qquad (3.1)$$

Proof. Consider the left-hand side of (3.1), we have

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x+1,y)+B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y+1)$$
$$=\int_0^1 \{u^x(1-u)^{y-1}+u^{x-1}(1-u)^y\}$$
$$\times E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du,$$

$$=\int_0^1 u^{x-1}(1-u)^{y-1}\{u+(1-u)\}$$
$$\times E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du,$$

$$=\int_0^1 u^{x-1}(1-u)^{y-1}\, E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du$$

which proves the desired result.

Theorem 3.2. The extension of Beta function satisfies the following summation formulas:

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,1-y)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x+n,1) \quad (3.2)$$

Proof. Consider the generalized binomial theorem

$$(1-u)^{-y}=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}u^n, \quad (|u|<1). \quad (3.3)$$

Applying (3.3) to the definition (2.1) of extended Beta function

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,1-y) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} u^{x+n-1} E^{k_3,k_4,q}_{k_1,k_2,p}\left(-\frac{\omega_1}{u} - \frac{\omega_2}{(1-u)}\right) du \cdot$$

Now, interchanging the order of summation and integration in above equation and using (2.1) proves the desired result.

Theorem 3.3. The extension of Beta function satisfies the following infinite summation
Formulas:

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,y) = \sum_{n=0}^{\infty} B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x+n,y+1) \cdot \quad (3.4)$$

Proof. Using the relation

$$(1-u)^{y-1} = (1-u)^y \sum_{n=0}^{\infty} u^n ,$$

In (2.1), we obtain

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,y) = \int_0^1 (1-u)^y \sum_{n=0}^{\infty} u^{x+n-1} E^{k_3,k_4,q}_{k_1,k_2,p}\left(-\frac{\omega_1}{u} - \frac{\omega_2}{(1-u)}\right) du \cdot$$

By interchanging the order of integration and summation in above equation and using (2.1), we get the desired result.

Theorem 3.4. The following relation holds true

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,y) = \sum_{m=0}^{n} \binom{n}{m} B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x+m,y+n-m), \qquad (3.5)$$

$$(n \in \mathbb{N}_0).$$

Proof. Using (3.1),

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,y) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x+1,y) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(x,y+1),$$

Taking $x = \alpha$ and $y = -\alpha - n$ in above relation, we have

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha-n) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha-n+1) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+1,-\alpha-n),$$

Starting with $n = 1, 2, 3, \ldots$, we have

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha-1) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+1,-\alpha-1),$$

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha-2) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha) + 2B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+1,-\alpha-1) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+2,-\alpha-2),$$

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha-3) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,-\alpha) + 3B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+1,-\alpha-1) + 3B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+2,-\alpha-2) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha+3,-\alpha-3)$$

and so on. The above series behaves like as finite binomials series does. Thus, we can finally obtain the desired relation (3.5). Note that, we can also prove the desired inequality by applying induction on n.

4. EXTENSION OF HYPERGEOMETRIC FUNCTIONS

In this section, we introduce further extension of hypergeometric function by using the extension of Beta function (2.1) as follows:

Definition 6.1. The extension of extended hypergeometric function is defined as:

$$F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} (\alpha)_n \frac{B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{(z)^n}{n!}, \quad (4.1)$$

$$(k_1,k_2,k_3,k_4 \in R_0^+, Re(\gamma) > Re(\beta) > 0, \omega_1,\omega_2 \geq 0, |z| < 1, p,q > 0, q < Re(k_1)+p).$$

Theorem 4.2. The extended hypergeometric has the following integral representation:

$$F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-\alpha} \times E^{k_3,k_4,q}_{k_1,k_2,p}\left(-\frac{\omega_1}{u} - \frac{\omega_2}{(1-u)}\right) du, \qquad (4.2)$$

$$(k_1,k_2,k_3,k_4 \in R_0^+, Re(\gamma) > Re(\beta) > 0, \omega_1,\omega_2 \geq 0, |z| < 1, p,q > 0, q < Re(k_1)+p).$$

Proof. By using (2.1) in (4.1), we have

$$F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1} \times E^{k_3,k_4,q}_{k_1,k_2,p}\left(-\frac{\omega_1}{u} - \frac{\omega_2}{(1-u)}\right) \sum_{n=0}^{\infty} (\alpha)_n \frac{(zu)^n}{n!} d\varphi,$$

Thus, by using, the relation

$$\sum_{n=0}^{\infty} (\alpha)_n \frac{(zu)^n}{n!} = (1-zu)^{-\alpha},$$

We get the desired relation.

Theorem 4.3. Consider $F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(.)$ function. Then, the following functional relation hold:

$$F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma;z) = \frac{\beta}{\gamma} F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta+1;\gamma+1;z) + \frac{\gamma-\beta}{\gamma} F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma+1;z), \qquad (4.3)$$

$$(k_1,k_2,k_3,k_4 \in R_0^+, Re(\gamma) > Re(\beta) > 0, \omega_1,\omega_2 \geq 0, |z| < 1, p,q > 0, q < Re(k_1)+p).$$

Proof. Using the following relation (3.1) in (4.1), we get

$$B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n+1,\gamma-\beta) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta+1) = B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta),$$

$$F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} (\alpha)_n \frac{\left(B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n+1,\gamma-\beta) + B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta+1)\right)}{B(\beta,\gamma-\beta)} \frac{z^n}{n!},$$

$$= \sum_{n=0}^{\infty} (\alpha)_n \frac{B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n+1,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (\alpha)_n \frac{B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta+1)}{B(\beta,\gamma-\beta)} \frac{z^n}{n!},$$

$$= \frac{B(\beta+1,\gamma-\beta)}{B(\beta,\gamma-\beta)} \sum_{n=0}^{\infty} (\alpha)_n \frac{B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n+1,\gamma-\beta)}{B(\beta+1,\gamma-\beta)} \frac{z^n}{n!} + \frac{B(\beta,\gamma-\beta+1)}{B(\beta,\gamma-\beta)} \sum_{n=0}^{\infty} (\alpha)_n \frac{B^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(\beta+n,\gamma-\beta+1)}{B(\beta,\gamma-\beta+1)} \frac{z^n}{n!}.$$

Then, using the value of Beta function in terms of gamma function together with (3.1), allow us to get the desired result.

Theorem 4.2. Consider $F^{k_3,k_4,q,\omega_2}_{k_1,k_2,p,\omega_1}(.)$ function. Then, the following sum relation hold:

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)=\sum_{m=0}^{\infty}\frac{(1+\beta-\gamma)_m}{m!}\frac{B(\beta+m,1)}{B(\beta,\gamma-\beta)}$$
$$\times F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta+m;\beta+m+1;z), \qquad (4.4)$$
$$(k_1,k_2,k_3,k_4\in R_0^+,Re(\gamma)>Re(\beta)>0,\omega_1,\omega_2\geq 0,|\varphi| < 1,p,q>0,q<Re(k_1)+p).$$

Proof. Using the following relation (3.2) in (4.1), we get

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+n,c-\beta)$$
$$=\textstyle\sum_{m=0}^{\infty}\frac{(1-\gamma+\beta)_m}{m!}B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+n+m,1)\cdot$$
$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}(\alpha)_n\frac{1}{B(\beta,\gamma-\beta)}$$
$$\times\textstyle\sum_{m=0}^{\infty}\frac{(1-\gamma+\beta)_m}{m!}B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+n+m,1)\frac{(z)^n}{n!},$$
$$=\textstyle\sum_{m=0}^{\infty}\frac{(1+\beta-\gamma)_m}{m!}\frac{B(\beta+m,1)}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+n+m,1)}{B(\beta+m,1)}\frac{(z)^n}{n!},$$

Then, using the definition of Gauss hypergeometric function (4.1), allow us to get the desired result.

Theorem 4.3. Consider $F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(.)$ function. Then, the following sum relation hold:

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)=(\gamma-\beta)$$
$$\times\textstyle\sum_{m=0}^{\infty}\frac{(\beta)_m}{(\gamma)_m}F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta+m;\gamma+m+1;z), \qquad (4.5)$$
$$(k_1,k_2,k_3,k_4\in R_0^+,Re(\gamma)>Re(\beta)>0,\omega_1,\omega_2\geq 0,|z| < 1,p,q>0,q<Re(k_1)+p).$$

Proof. Using the following relation (3.4) in (6.1), we get

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)=\sum_{m=0}^{\infty}B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x+m,y+1)\cdot$$
$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)$$
$$=\sum_{n=0}^{\infty}(\alpha)_n\sum_{m=0}^{\infty}\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+m+n,\gamma-\beta+1)}{B(\beta,\gamma-\beta)}\frac{(z)^n}{n!},$$
$$=\textstyle\sum_{m=0}^{\infty}\frac{B(\beta+m,\gamma-\beta+1)}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+m+n,\gamma-\beta+1)}{B(\beta+m,\gamma-\beta+1)}\frac{(z)^n}{n!}.$$

Then, using the value of Beta function in terms of gamma function together with (3.1), allow us to get the desired result.

Theorem 4.4. Consider $F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(.)$ function. Then, the following result hold:

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)=\sum_{m=0}^{n}\binom{n}{m}\frac{B(\beta+m,\gamma-\beta+n-m)}{B(\beta,\gamma-\beta)}$$
$$\times F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta+m;\gamma+n;z), \qquad (4.6)$$
$$(k_1,k_2,k_3,k_4\in R_0^+,Re(\gamma)>Re(\beta)>0,\omega_1,\omega_2\geq 0,|z| < 1,p,q>0,q<Re(k_1)+p).$$

Proof. Using the following relation (3.5) in (4.1), we get

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)$$
$$=\textstyle\sum_{m=0}^{n}\binom{n}{m}B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x+m,y+n-m),$$
$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)$$
$$=\sum_{r=0}^{\infty}(\alpha)_r\sum_{m=0}^{n}\binom{n}{m}\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+r+m,\gamma-\beta+n-m)}{B(\beta,\gamma-\beta)}\frac{(z)^r}{r!},$$
$$=\sum_{m=0}^{n}\binom{n}{m}\frac{B(\beta+m,\gamma-\beta+n-m)}{B(\beta,\gamma-\beta)}\sum_{r=0}^{\infty}(\alpha)_r$$
$$\times\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta+r+m,\gamma-\beta+n-m)}{B(\beta+m,\gamma-\beta+n-m)}\frac{(z)^r}{r!},$$

Then, using the definition of Gauss hypergeometric function (4.1), allow us to get the desired result**.**

5. TRANSFORMATION AND SUMMATION FORMULAS

In this section, we obtain transformation and summation formulas for the extended hypergeometric function as follows:

Theorem 5.1. The following transformation for extended hypergeometric function holds
true for $\omega_1,\omega_2\geq 0$;

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)$$
$$=(1-z)^{-\alpha}F_{k_1,k_2,p,\omega_2}^{k_3,k_4,q,\omega_1}\left(\alpha,\gamma-\beta;\gamma;\tfrac{-z}{1-z}\right), \qquad (5.1)$$
$$(k_1,k_2,k_3,k_4\in R_0^+,Re(\gamma)>Re(\beta)>0,\omega_1,\omega_2\geq 0,|z| < 1,p,q>0,q<Re(k_1)+p).$$

Proof. Replacing $u\longrightarrow 1-u$ in (4.2) and using the following result:

$$\left(1-z(1-u)\right)^{-\alpha}=(1-z)^{-\alpha}\left(1-\frac{z}{1-z}u\right)^{-\alpha}, \qquad (5.2)$$

We obtain

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;z)$$
$$=\frac{(1-\varphi)^{-\alpha}}{B(\beta,\gamma-\beta)}\int_0^1u^{\gamma-\beta-1}(1-u)^{\beta-1}\left(1-\frac{z}{1-z}u\right)^{-a}$$
$$\times E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du\cdot$$

Which, by applying (4.1) yields the desired result.

Theorem 5.2. The following summation formula holds true:

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;1)=\frac{B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\beta,\gamma-\alpha-\beta)}{B(\beta,\gamma-\beta)}, \qquad (5.3)$$
$$\left(\begin{array}{c}k_1,k_2,k_3,k_4\in R_0^+,Re(\gamma)>Re(\beta)>0,Re(\gamma-\alpha-\beta)>\\ 0\ ,\omega_1,\omega_2\geq 0,p,q>0,\\ |z|<1,q<Re(k_1)+p\end{array}\right)$$

Proof. Taking $\varphi=1$ in (4.2), we have

$$F_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(\alpha,\beta;\gamma;1)=\frac{1}{B(\beta,\gamma-\beta)}$$
$$\times\textstyle\int_0^1u^{\beta-1}(1-u)^{\gamma-\alpha-\beta-1}E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)du\cdot$$

By applying definition (2.1) to the above equation, we get the desired result.

6. CONNECTION WITH THE OTHER SPECIAL FUNCTIONS

In this section we obtain certain connections of the generalized Beta function (2.1) in terms of other functions and polynomials. The results obtained here are interesting and can further be applied to other extensions of beta and other functions.

Theorem 6.1. The following relationship further generalized extended Beta function between the Wright function holds true:

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)=\frac{\Gamma(k_4)}{\Gamma(k_3)}\int_0^1u^{x-1}(1-u)^{y-1}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(k_3,q),(1,1);\\(k_4,q),(k_1,k_2);\end{array}-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right]du. \qquad (6.1)$$

Proof: Applying the result from Salim et al., (see [25], (29))

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z)=\frac{\Gamma(k_4)}{\Gamma(k_3)}\,{}_2\psi_2\left[\begin{matrix}(k_3,q),(1,1);\\(k_4,q),(k_1,k_2);\end{matrix}z\right]. \qquad (6.2)$$

On setting $z\longrightarrow-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}$, multiplying both sides by $u^{x-1}(1-u)^{y-1}$and integrating with respect to u limit from 0 to 1, gives the desired result.

Theorem 6.2. The following relationship further generalized extended Beta function between the Fox's H-function holds true:

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)=\frac{\Gamma(k_4)}{\Gamma(k_3)}\int_0^1 u^{x-1}(1-u)^{y-1}$$

$$\times H_{2,3}^{1,2}\left[\frac{\omega_1}{u}+\frac{\omega_2}{(1-u)}\middle|\begin{matrix}(0,1), & (1-k_3,q)\\(0,1),(1-k_2,k_1),(1-k_4,p)\end{matrix}\right]du. \qquad (6.3)$$

Proof: Applying the result from Salim et al., (see [25], (32))

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z)$$

$$=\frac{\Gamma(k_4)}{\Gamma(k_3)}H_{2,3}^{1,2}\left[-z\middle|\begin{matrix}(0,1), & (1-k_3,q)\\(0,1),(1-k_2,k_1),(1-k_4,p)\end{matrix}\right]. \qquad (6.4)$$

On setting $z\longrightarrow-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}$, multiplying both sides by $u^{x-1}(1-u)^{y-1}$ and integrating with respect to t limit from 0 to 1, gives the desired result.

Theorem 6.3. The following relationship further generalized extended Beta function between the Generalized hypergeometric function holds true:

$$B_{k_1,k_2,p,\omega_1}^{k_3,k_4,q,\omega_2}(x,y)=\frac{1}{\Gamma(k_2)}\int_0^1 u^{x-1}(1-u)^{y-1}$$

$$\times\,{}_{q+1}F_{p+n}\left[\begin{matrix}1 & ,\Delta(q,k_3)\\\Delta(n,k_2) & ,(p,k_4)\end{matrix};\left(-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}\right)\frac{q^q}{p^p n^n}\right]$$

$$\times du, \qquad (6.5)$$

Where $k_1=n\in\mathbb{N}$, $\Delta(n,k)$is k −tuple $\frac{n}{k},\frac{n+1}{k},\dots,\frac{n+k-1}{k}$.

Proof: Applying the result from Salim et al., (see [25], (30))

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z)$$

$$=\frac{1}{\Gamma(k_2)}\,{}_{q+1}F_{p+n}\left[\begin{matrix}1,\ \Delta(q,k_3)\\\Delta(n,k_2),\ \Delta(p,k_4)\end{matrix};\frac{zq^q}{p^p n^n}\right]. \qquad (6.6)$$

On setting $z\longrightarrow-\frac{\omega_1}{u}-\frac{\omega_2}{(1-u)}$, multiplying both sides by $u^{x-1}(1-u)^{y-1}$and integrating with respect to u limit from 0 to 1, gives the desired result.

7. CONCLUSIONS

This study introduced a new extension of the extended Beta and Gauss hypergeometric functions, exploring their properties, integral representations, and connections with other special functions. These findings provide a foundation for further research into broader generalizations and applications in mathematical and applied sciences.

REFERENCES

[1] U. M. Abubakar and S. R. Kabara, "New generalized extended Gamma and Beta functions with their applications," *International Scientific Research and Innovation Congress*, pp. 446–465, 2021.

[2] A. A. Al-Gonah and W. K. Mohammed, "A new extension of extended Gamma and Beta functions and their properties," *Journal of Scientific and Engineering Research*, vol. 5, no. 9, pp. 257–270, 2018.

[3] A. A. Al-Gonah and W. K. Mohammed, "A new form of extended hypergeometric function and their properties," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 30–41, 2021. Available: https://1030538/psp-eas/2021.0059.

[4] M. Ali, "A further extension of Beta and Related functions," *Palestine Journal of Mathematics*, vol. 12, no. 3, pp. 65–73, 2023.

[5] A. A. Atash, S. S. Barahmah, and M. A. Kulib, "On a new extension of extended Gamma and Beta functions," *International Journal of Statistics and Applied Mathematics*, vol. 3, no. 6, pp. 14–18, 2018.

[6] A. A. Atash, S. S. Barahmah, and M. A. Kulib, "On extensions of extended Gauss hypergeometric function," *Communications in Advanced Mathematical Sciences*, vol. 2, no. 3, pp. 199–205, 2019.

[7] N. Dudi and U. M. Abubakar, "A further extension of Gamma and Beta functions involving generalized Mittag-Leffler function and its applications," *Journal of Physical Sciences*, vol. 26, pp. 61–71, 2021.

[8] A. Enes and İ. O. K., "Generalized gamma, beta and hypergeometric functions defined by Wright function and applications to fractional differential equations," *Cumhuriyet Science Journal*, vol. 43, no. 4, pp. 684–695, 2022.

[9] K. S. Gehlot and K. S. Nisar, "Extension of two-parameter Gamma, Beta functions and its properties," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 15, no. 3, pp. 39–55, 2020.

[10] R. Goyal, S. Momani, P. Agarwal, and M. T. Rasiaas, "An extension of Beta function by using Wiman's function," *Axiom*, vol. 10, no. 187, pp. 1–11, 2021. Available: http://dx.doi.org/10.3390/axiom1003087.

[11] S. R. Kabara, A. H. Abdullahi, M. A. Lawan, F. A. Idris, M. S. Musa, and S. I. Musa, "Generalized extended Beta function," *International Journal of Applied Science and Research*, vol. 6, pp. 2581–7876, 2023.

[12] N. Khan and S. Husain, "A note on extended Beta function involving generalized Mittag-Leffler function and its applications," *TWMS Journal of Applied and Engineering Mathematics*, vol. 12, no. 1, pp. 71–81, 2022.

[13] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Applications*. London, New York: Chapman and Hall/CRC, 2004.

[14] M. A. Kulib and A. A. Al-Gonah, "New extended Beta function defined by product of two Wright functions," *International Journal of Mathematics Modelling & Computations*, vol. 15, no. 1, pp. 21–27, 2025.

[15] M. A. Kulib, F. B. F. Mohsen, and S. S. Barahmah, "Further extended Gamma and Beta functions in terms of generalized Wright function," *Electronic Journal of University of Aden for Basic and Applied Sciences*, vol. 1, no. 2, pp. 77–83, 2020.

https://doi.org/10.20428/jst.v30i5.2838

[16] . M. A. Kulib, A. A. Al-Gonah, and S. S. Barahmah, “Certain investigations in the field of special functions,” *Journal of Mathematical Analysis and Modeling*, vol. 1, no. 1, pp. 87–98, 2020.

[17] M. J. Luo, G. V. Milovanovic, and P. Agarwal, “Some results on the extended beta and extended hypergeometric functions,” *Applied Mathematics and Computation*, vol. 248, pp. 631–651, 2014.

[18] G. M. Mittag-Leffler, “Sur la nouvelle fonction Eα(x)E_{\alpha}(x)Eα(x),” *C. R. Acad. Sci. Paris*, vol. 137, pp. 554–558, 1903.

[19] K. Oraby, A. Rizq, M. Ahmed, M. Khaled, E. Ahmed, and M. Magdy, “Generalization of Beta functions in terms of Mittag-Leffler function,” *Frontiers in Scientific Research and Technology*, vol. 1, pp. 81–88, 2020.

[20] M. A. Ozarslan and B. Yılmaz, “The extended Mittag-Leffler function and its properties,” *Journal of Inequalities and Applications*, vol. 2014, no. 1, p. 85, 2014.

[21] T. R. Prabhakar, “A singular integral equation with a generalized Mittag-Leffler function in the kernel,” *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[22] E. D. Rainville, *Special Functions*, New York, NY, USA: Chelsea Publishing Company, 1960.

[23] M. K. Rasheed and A. H. Majeed, “Generalization of gamma and beta functions with certain properties and statistical applications,” *Iraqi Journal of Science*, vol. 64, no. 7, pp. 4487–4495, 2023.

[24] T. O. Salim, “Some properties relating to the generalized Mittag-Leffler function,” *Advances in Mathematics Analysis*, vol. 4, pp. 21–23, 2009.

[25] T. O. Salim and A. W. Faraj, “A generalization of Mittag-Leffler function and integral operator associated with fractional calculus,” *Journal of Fractional Calculus and Applications*, vol. 3, no. 5, pp. 1–13, 2012.

[26] M. Shadab, S. Jabee, and J. Choi, “An extension of beta function and its application,” *Far East Journal of Mathematical Sciences*, vol. 103, no. 1, pp. 235–251, 2018.

[27] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Chichester, UK: Halsted Press (Ellis Horwood Limited), John Wiley & Sons, 1984.