# DECOMPOSITION OF GENERALIZED RECURRENT TENSOR FIELDS OF R^H-NTH ORDER IN FINSLER MANIFOLDS

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# Decomposition of Generalized Recurrent Tensor Fields of R^h-nth Order in Finsler Manifolds

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***Abstract***— In this paper, we investigate the decomposition of Cartan's third curvature tensor $\boldsymbol{R^i_{jkh}}$ in the framework of generalized $\boldsymbol{R^h}$-nth recurrent Finsler spaces. We examine three distinct decompositions of the curvature tensor and analyze the resulting equations under covariant derivatives of the decomposed tensor fields. The decompositions are expressed in terms of independent tensor fields, and we explore the recurrence properties of the resulting decomposition tensors. We show that under certain conditions, these decomposition tensors exhibit generalized nth-recurrence properties, which are crucial for understanding the geometric behavior of these tensors in Finsler geometry. The results provide a deeper understanding of the curvature properties in higher-dimensional recurrent Finsler spaces, with implications for both theoretical and applied geometry.

**Keywords—** Finsler Manifold, Generalized Recurrent Tensor Field, Decomposition.

## I. INTRODUCTION

The study of curvature tensors in differential geometry is vital for understanding the structure of various geometric spaces, particularly Finsler spaces. Finsler spaces, which generalize Riemannian manifolds, provide a natural framework for examining various curvature properties. Cartan's third curvature tensor, denoted as $R^i_{jkh}$ , plays a crucial role in describing the geometry of such spaces. In this paper, we explore the decomposition of this curvature tensor in the context of generalized $R^h$ -nth recurrent Finsler spaces. Specifically, we discuss how the curvature tensor can be decomposed into various forms, and we investigate the behavior of these decompositions under certain conditions. Through these decompositions, we derive recurrence properties of the corresponding tensors, which are essential for understanding the structure of these spaces in higher-dimensional settings.

Tensor fields play a fundamental role in differential geometry, serving as essential tools for describing geometric objects and their properties. In particular, recurrent tensor fields, characterized by a covariant derivative that is proportional to the tensor itself, have been extensively studied in the context of Riemannian and pseudo-Riemannian manifolds. However, the study of recurrent tensor fields in Finsler manifolds, which are more general geometric structures than Riemannian manifolds, remains a relatively unexplored area.

In this paper, we focus on a specific class of tensor fields known as generalized recurrent tensor fields of R-nth order. These tensor fields exhibit a more complex recurrence relation compared to standard recurrent tensor fields. Our primary objective is to investigate the decomposition properties of these tensor fields within the framework of Finsler geometry. By decomposing a generalized recurrent tensor field into simpler components, we aim to gain deeper insights into its geometric significance and to establish a more comprehensive theory of tensor fields in Finsler manifolds. Finsler manifolds are a generalization of Riemannian manifolds where the metric tensor depends not only on the point of the manifold but also on the direction of the tangent vector at that point. This allows for a more general study of geometry and physics in spaces with non-constant curvature. Generalized recurrent tensor fields are tensor fields that satisfy a certain recurrence relation. This relation can be used to study the geometry and dynamics of Finsler manifolds.

In this paper, we will discuss the decomposition of generalized recurrent tensor fields of -nth order in Finsler manifolds. We will obtain different tensors which satisfy the recurrence property under the decomposition. Also, we will prove the decomposition for different tensors are non-vanishing. As an illustration of the applicability of the obtained results, we will finish this work with some illustrative examples. This paper is organized as follows. In Section 2, we will introduce the necessary preliminaries. In Section 3, we will discuss the decomposition of generalized recurrent tensor fields of nth order. In Section 4, we will give some illustrative examples. Finally, in Section 5, we will give some concluding remarks. The study of curvature tensors in Finsler geometry has been an area of significant research in recent years, as it provides deep insights into the structural properties of Finsler spaces. Various forms of curvature decomposition have been explored to better understand the geometrical behavior of these spaces [1] introduced certain types of generalized recurrent Finsler spaces, contributing to the development of the field. Subsequent works, such as those by [2, 3, 4, 5, 6, 7], extended the concept of generalized Finsler spaces by analyzing special curvature tensors and their recurrence properties, which are pivotal in understanding higher-order generalizations of Finsler structures. In addition, the decomposition of curvature tensors, a technique pioneered by researchers like [8, 9, 10] has been used to examine the impact of covariant derivatives on the structure of Finsler spaces.

These contributions provide a comprehensive framework for studying the complex interactions between curvature tensors and space recurrence in various Finsler settings. Moreover, works by [11, 12] highlight the significance of these decompositions for exploring advanced geometrical structures in both theoretical and applied contexts. This paper builds upon these prior developments to further investigate the decompositions of Cartan's third curvature tensor in generalized Finsler spaces, thereby enriching the understanding of curvature properties in higher-dimensional recurrent Finsler spaces.

## II. PRELIMINARIES

É. Cartan in his second postulate, represented the variation of an arbitrary vector field $X^i$ under the infinitesimal change of its line element $(x\,, y)$ to $(x+dx\,, y+dy)$ by means of covariant (absolute) differential given by

(2.1) $DX^i = dX^i + X^j\left(C^i_{jk}\, dy^k + \Gamma^i_{jk}\, dx^k\right)$, where

(2.2) a) $\Gamma^i_{jk} = \gamma^i_{jk} - C^i_{mk} G^m_j + g^{ih} C_{jkm}\, G^m_h$

b) $G^i = \frac{1}{2}\, \gamma^i_{jk}\, y^j\, y^k$ and c) $G^i_j = \dot{\partial}_j\, G^i$

The function $G^i$ is positively homogeneous of degree two in the directional argument.

Eliminating $dy^k$ from (2.1) and in terms of the absolute differential of $l^i$, É. Cartan deduced

(2.3) $DX^i = F\, X^i|_k\, D\, l^k + X^i_{|k}\, dx^k + y^k\left(\dot{\partial}_k\, X^i\right)\frac{dF}{F}$,

where,

(2.4) a) $X^i|_k = \dot{\partial}_k\, X^i + X^r\, C^i_{rk}$,

b) $X^i_{|k} = \partial_k\, X^i + X^r\, \Gamma^{*i}_{rk} - \left(\dot{\partial}_m\, X^i\right)\Gamma^m_{sk}\, y^s$, and

c) $\Gamma^{*i}_{rk} = \Gamma^i_{rk} - C^i_{mr}\, \Gamma^m_{sk}\, y^s$

The function $\Gamma^{*i}_{rk}$ defined by (2.4c) is connection parameter of É. Cartan, this is symmetric in the lower indices $r$ and $k$ and positively homogeneous of degree zero in the directional argument and satisfies:

(2.5) $g_{ih}\, \Gamma^{*i}_{rk} = \Gamma^*_{rhk}$

The equations (2.4a) and (2.4b) give two processes of covariant differentiation called v-covariant differentiation (Cartan's first kind covariant differentiation) and h-covariant differentiation (Cartan's second kind covariant differentiation), respectively. So $X^i|_k$ and $X^i_{|k}$ are respectively $v$-covariant derivative an d h -covariant derivative of the vector field $X^i$. The metric tensor $g_{ij}$ and the associate metric tensor $g^{ij}$ are related by

(2.6) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k\,, \\ 0 & \text{if } i \neq k\,. \end{cases}$

The quantities $g_{ij}$ , $g^{ij}$ and $\delta^i_j$ are satisfies

(2.7) a) $g_{ij}\, g^{ij} = n$ and b) $\delta^i_j\, g_{ik} = g_{jk}$ .

The vector $y_i$ satisfies relation

(2.8) $y_i\, y^i = F^2$

The vectors $y_i$ and $\delta^i_k$ also satisfy the following relations

(2.9) a) $\delta^i_k\, y^k = y^i$

b) $\delta^i_j\, g^{jk} = g^{ik}$ and

c) $g_{ij}\, y^j = y_i$

The metric tensor $g_{ij}$ and the associate metric tensor $g^{ij}$ are covariant constant with respect to both processes

(2.10) a) $g_{ij|m} = 0$ and b) $g^{ij}_{|m} = 0$

The vectors $y^i$ , $y_i$ are vanish under h-covariant differentiation

(2.11) a) $y_{i|m} = 0$ and b) $y^i_{|m} = 0$

The h-curvature tensor $R^i_{jkh}$ (Cartan's third curvature tensor), is defined by

(2.12) $R^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + \left(\dot{\partial}_l\, \Gamma^{*i}_{jk}\right) G^l_h + C^i_{jm}\left(\dot{\partial}_k\, G^m_h - G^m_{kl}\, G^l_h\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - [\dot{\partial}_k\, \Gamma^{*i}_{jh} + \left(\dot{\partial}_l\, \Gamma^{*i}_{jh}\right) G^l_k + C^i_{jm}\left(\dot{\partial}_h\, G^m_k - G^m_{hl}\, G^l_k\right) + \Gamma^{*i}_{mh}\, \Gamma^{*m}_{jk}]$

The h-curvature tensor $R^i_{jkh}$ is positively homogeneous of degree $-1$ in the directional argument and skew-symmetric in the last two lower indices $h$ and $k$ , i.e.

(2.13) $R^i_{jkh} = -\, R^i_{jhk}$

**Definition 2.1:** Let the current coordinates in the tangent space at the point $x_0$ to $x^i$. Then the indicatrix $I_{n-1}$ is a hypersurface defined by [5] $F(x_0, x^i) = 1$ or by the parametric defined by $x^i = x^i(u^a), a = 1,2, \dots, n-1$.

**Definition 2.2:** The projection of any tensor $T^i_j$ on indicatrix $I_{n-1}$ given by [5]

(2.14) $pT^i_j = T^a_b h^i_a h^b_j$ , where

(2.15) $h^i_c = \delta^i_c - l^i l_c$ .

Let us consider a Finsler space $F_n$ whose Cartan's third curvature tensor $R^i_{jkh}$ satisfies the following condition

(2.16) $R^i_{jkh|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n}\, R^i_{jkh} + \mu_{m_1 m_2 \dots m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$ ,

where $R^i_{jkh} \neq 0$ and $|m_1|m_2| \dots |m_n$ are h-covariant differentiation (Cartan's second kind covariant differential operator) with respect to $x^m$ to nth order, $\lambda_{m_1 m_2 \dots m_n}$ and $\mu_{m_1 m_2 \dots m_n}$ are recurrence tensors fields.

**Definition 2.3:** A Finsler space $F_n$ whose Cartan's third curvature tensor $R^i_{jkh}$ satisfying the condition (2.16), where $\lambda_{m_1 m_2 \dots m_n}$ and $\mu_{m_1 m_2 \dots m_n}$ are non-null covariant tensors fields, is called a generalized $R^h$-nth order space and the tensor will be called generalized h-nth tensor. We shall denote this space briefly by $GR^h$-$n^{th}RF_n$ .

## III. Decomposition of Curvature Tensor Field $R^i_{jkh}$

In this paper, we will discuss the decomposition of Cartan's third Curvature tensor $R^i_{jkh}$ in generalized $R^h$-nth recurrent Finsler Space. We consider the decomposition of curvature tensor field $R^i_{jkh}$ in the following way [7]

(3.1) $R^i_{jkh} = U^i\, V_{jkh}$ ,

(3.2) $R^i_{jkh} = U_k V^i_{jh}$ and

(3.3) $R^i_{jkh} = U^i_j\, V_{kh}$ .

Further considering the decomposition of the Curvature tensor $R^i_{jkh}$ in the form [6]

(3.4) $R^i_{jkh} = V^i_j \emptyset_k \psi_h$ .

Let us consider Cartan's third Curvature tensor $R^i_{jkh}$ is decomposable as (3.1), where $V_{jkh}$ is non-zero covariant tensor field which called decomposition tensor field and $U^i$ is independent of $x^m$ to nth order.

Taking h-covariant derivative of (3.1) with respect to $x^m$ to nth order, assume that $U^i$ is constant we get:

$$R^i_{jkh|m_1|m_2|m_3|\dots|m_n} = U^i V_{jkh|m_1|m_2|m_3|\dots|m_n} .$$

By using the condition (2.16) in above equation and in view of (3.1), we get

(3.5) $U^i V_{jkh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} U^i\, V_{jkh} + \mu_{m_1 m_2 m_3\dots m_n}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$ .

Since $U^i$ is independent of $x^m$ to nth order, equation (3.5) can be written as

$$(U^i V_{jkh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n}(U^i V_{jkh}) + \mu_{m_1 m_2 m_3\dots m_n}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}).$$

Therefore, the proof of theorem is completed, we can say,

**Theorem 3.1.** In $GR^h - n^{th}RF$, under the decomposition (3.1) and if $U^i$ is constant, then the decomposition tensor $(U^i\, V_{jkh})$ satisfies the generalized nth-recurrence property.

In view of equation (3.5), and where $\alpha_{m_1 m_2 m_3\dots m_n i} = \mu_{m_1 m_2 m_3\dots m_n}/U^i$ we get:

(3.6) $V_{jkh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V_{jkh} + \alpha_{m_1 m_2 m_3\dots m_n i}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$.

Above equation can be written as

(3.7) $V_{jkh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V_{jkh} + (\emptyset_{m_1 m_2 m_3\dots m_n hjk} - \emptyset_{m_1 m_2 m_3\dots m_n kjh})$ ,

where $\emptyset_{m_1 m_2 m_3\dots m_n hjk} = \alpha_{m_1 m_2 m_3\dots m_n i}\delta^i_h\, g_{jk}$ and $\emptyset_{m_1 m_2 m_3\dots m_n kjh} = \alpha_{m_1 m_2 m_3\dots m_n i}\delta^i_k\, g_{jh}$ .

Now, if the tensor field $\emptyset_{m_1 m_2 m_3\dots m_n hjk} = -\emptyset_{m_1 m_2 m_3\dots m_n kjh}$ , then above equation can be written as

(3.8) $V_{jkh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V_{jkh} + 2\emptyset_{m_1 m_2 m_3\dots m_n hjk}$ .

**Corollary 3.1.** In $GR^h - n^{th}RF$, under the decomposition (3.1) , the decomposition tensor $(V_{jkh})$ is non-vanishing .

Let us consider Cartan's third Curvature tensor $R^i_{jkh}$ is decomposable as (3.2), where $V^i_{jh}$ is decomposition tensor field and $U_k$ is independent of $x^m$ to nth order.

Taking h-covariant derivative of (3.2) with respect to $x^m$ to nth order, assume that $U_k$ is covariant constant we get

$$R^i_{jkh|m_1|m_2|m_3|\dots|m_n} = U_k V^i_{jh|m_1|m_2|m_3|\dots|m_n} .$$

By using the condition (2.16) in above equation and in view of (3.2), we get

(3.9) $U_k V^i_{jh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} U_k V^i_{jh} + \mu_{m_1 m_2 m_3\dots m_n}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$.

Since $U_k$ is independent of $x^m$ to nth order, equation (3.9) can be written as

(3.10) $(U_k V^i_{jh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n}(U_k V^i_{jh}) + \mu_{m_1 m_2 m_3\dots m_n}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$

So, the proof of theorem is completed, we can say,

**Theorem 3.2.** In $GR^h - n^{th}RF$, under the decomposition (3.2) and if $U_k$ is covariant constant, then the decomposition tensor $(U_k V^i_{jh})$ satisfies the generalized nth-recurrence property.

In view of equation (3.9), and where $\omega^k_{m_1 m_2 m_3\dots m_n} = \mu_{m_1 m_2 m_3\dots m_n}/U_k$ we get

(3.11) $V^i_{jh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V^i_{jh} + \omega^k_{m_1 m_2 m_3\dots m_n}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$.

Above equation can be written as

(3.12) $V^i_{jh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V^i_{jh} + (\theta^i_{m_1 m_2 m_3\dots m_n hj} - \theta^i_{m_1 m_2 m_3\dots m_n jh})$ ,

where $\theta^i_{m_1 m_2 m_3\dots m_n hj} = \omega^k_{m_1 m_2 m_3\dots m_n}\delta^i_h\, g_{jk}$ and $\theta^i_{m_1 m_2 m_3\dots m_n jh} = \omega^k_{m_1 m_2 m_3\dots m_n}\delta^i_k\, g_{jh}$ . If the tensor field $\theta^i_{m_1 m_2 m_3\dots m_n hj}$ is symmetric in last two indicator, then the equation (3.12) can be written as

(3.13) $V^i_{jh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V^i_{jh}$ .

Transvecting (3.13) by $y^j$ using (2.11b), we get

(3.14) $V^i_{h|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n} V^i_h$ , where $V^i_h = V^i_{jh} y^j$ .

Transvecting (3.10) by $y^k$ using (2.11b), (2.9a) and (2.9c), we get

(3.15) $(UV^i_{jh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1 m_2 m_3\dots m_n}(UV^i_{jh}) + \mu_{m_1 m_2 m_3\dots m_n}(\, \delta^i_h\, y_j - y^i g_{jh}\,)$ ,

where $U = U_k y^k$ .

Transvecting (3.15) by $y^j$ using (2.11b), (2.8) and (2.9c), we get

(3.16) $(UV_h^i)_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(UV_h^i) + \mu_{m_1m_2m_3\dots m_n}(\delta_h^i F^2 - y^i y_h)$ ,

where $V_h^i = V_{jh}^i y^j$.

Contracting the indices i and h in (3.16) using (2.6) and (2.8), we get:

(3.17) $(UV)_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(UV) + \mu_{m_1m_2m_3\dots m_n}(n-1)F^2$, where $V_i^i = V$.

If $n = 1$ , the equation (3.17) can be written as

(3.18) $(UV)_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(UV)$

Since $U$ is constant, above equation can be written as

(3.19) $V_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}V$

**Corollary 3.2.** Under the decomposition (3.2), if $U_k$ is covariant constant and $U$ is constant, then the behavior of decomposition tensors $V_{jh}^i$ ,$V_h^i$ ,$UV$ and $V$ are nth-recurrent.

Let us consider Cartan's third Curvature tensor $R_{jkh}^i$ is decomposable as (3.3), where $U_j^i$ and $V_{kh}$ are the decomposition tensors field. Taking h-covariant derivative of (3.3) with respect to $x^m$ to nth order, assume that $U_j^i$ is independent of $x^m$ , we get

$$R^i_{jkh|m_1|m_2|m_3|\dots|m_n} = U_j^i V_{kh|m_1|m_2|m_3|\dots|m_n} .$$

By using the condition (2.16) in above equation and in view of (3.3), we get in above equation, we get

(3.20) $U_j^i V_{kh|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n} U_j^i V_{kh} + \mu_{m_1m_2m_3\dots m_n}(\delta_h^i g_{jk} - \delta_k^i g_{jh})$.

Since $U_j^i$ is independent of $x^m$ to nth order, equation (3.20) can be written as

(3.21) $(U_j^i V_{kh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(U_j^i V_{kh}) + \mu_{m_1m_2m_3\dots m_n}(\delta_h^i g_{jk} - \delta_k^i g_{jh})$ .

So, the proof of theorem is completed, we can say,

**Theorem 3.3.** In $GR^h - n^{th}RF$, under the decomposition (3.3), the decomposition tensor $(U_j^i V_{kh})$ satisfies the generalized nth-recurrence property. We can conclude that if $\delta_h^i g_{jk} = \delta_k^i g_{jh}$ , then the decomposition tensors $(U^i V_{jkh})$ , $(U_k V_{jh}^i)$ and $(U_j^i V_{kh})$ behave as nth-recurrent satisfy the following conditions

(3.22) $(U^i V_{jkh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(U^i V_{jkh})$ ,

(3.23) $(U_k V_{jh}^i)_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(U_k V_{jh}^i)$

and

(3.24) $(U_j^i V_{kh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(U_j^i V_{kh})$ ,

respectively.

**Example:** In order to illustrate the effectiveness of the proposed findings, we consider example of the nth-recurrence properties.

The decomposition tensor $(U^i V_{jkh})$ is nth-recurrent if and only if it satisfies

$$(U^i V_{jkh})_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}(U^i V_{jkh}) .$$

Firstly, since the decomposition tensor $(U^i V_{jkh})$ is nth-recurrent, the condition (3.22) is satisfied. In view of (2.14), the decomposition tensor $(U^i V_{jkh})$ on indicatrix given by

(3.25) $p.(U^i V_{jkh}) = U^a V_{bcd} h_a^i h_j^b h_k^c h_h^d$ .

By taking h-covariant derivative of (3.25) with respect to $x^m$ to nth order, using equation (3.22) and the fact that $h_b^a$ is constant in above equation, we get

$$[p.(U^i V_{jkh})]_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n} U^a V_{bcd} h_a^i h_j^b h_k^c h_h^d .$$

Using (3.25) in above equation, we get

(3.26) $[p.(U^i V_{jkh})]_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n}[p.(U^i V_{jkh})]$ .

Above equation means the projection on indicatrix for the decomposition tensor $(U^i V_{jkh})$ behaves as nth-recurrent. Secondly, let the projection on indicatrix for the decomposition tensor $(U^i V_{jkh})$ is nth-recurrent, that is, it satisfies equation (3.26). By using (3.25) in equation (3.26), we get

$$(U^a V_{bcd} h_a^i h_j^b h_k^c h_h^d)_{|m_1|m_2|m_3|\dots|m_n} = \lambda_{m_1m_2m_3\dots m_n} U^a V_{bcd} h_a^i h_j^b h_k^c h_h^d .$$

Since $h_j^i = \delta_j^i - l^i l_j$ , above equation we can written as

$$\begin{aligned}
&[(U^i V_{jkh}) - (U^i V_{jkd}) l^d l_h - (U^i V_{jch}) l^c l_k + (U^i V_{jcd}) l^c l_k l^d l_h - (U^i V_{bkh}) l^b l_j + (U^i V_{bkd}) l^b l_j l^d l_h + (U^i V_{bch}) l^b l_j l^c l_k - (U^i V_{bcd}) l^b l_j l^c l_k l^d l_h - (U^a V_{jkh}) l^i l_a + \\
&(U^a V_{jkd}) l^i l_a l^d l_h + (U^a V_{jch}) l^i l_a l^c l_k - (U^a V_{jcd}) l^i l_a l^c l_k l^d l_h + (U^a V_{bkh}) l^i l_a l^b l_j - (U^a V_{bkd}) l^i l_a l^b l_j l^d l_h - (U^a V_{bch}) l^i l_a l^b l_j l^c l_k + \\
&(U^a V_{bcd}) l^i l_a l^b l_j l^c l_k l^d l_h]_{|m_1|m_2|m_3|\dots|m_n} = \\
&\lambda_{m_1m_2m_3\dots m_n}[(U^i V_{jkh}) - (U^i V_{jkd}) l^d l_h \\
&\quad -(U^i V_{jch}) l^c l_k + (U^i V_{jcd}) l^c l_k l^d l_h - (U^i V_{bkh}) l^b l_j + (U^i V_{bkd}) l^b l_j l^d l_h \\
&\quad +(U^i V_{bch}) l^b l_j l^c l_k - (U^i V_{bcd}) l^b l_j l^c l_k l^d l_h - (U^a V_{jkh}) l^i l_a + (U^a V_{jkd}) l^i l_a l^d l_h \\
&\quad +(U^a V_{jch}) l^i l_a l^c l_k - (U^a V_{jcd}) l^i l_a l^c l_k l^d l_h + (U^a V_{bkh}) l^i l_a l^b l_j
\end{aligned}$$

$-(U^a V_{bkd}) l^i l_a l^b l_j l^d l_h - (U^a V_{bch}) l^i l_a l^b l_j l^c l_k + (U^a V_{bcd}) l^i l_a l^b l_j l^c l_k l^d l_h]$.

From, $l^i = \frac{y^i}{F}$ and $l_i = \frac{y_i}{F}$, if $(U^a V_{bcd}) y_a = (U^a V_{bcd}) y^b = (U^a V_{bcd}) y^c = (U^a V_{bcd}) y^d = 0$, then above equation can be written as

$$(U^i\, V_{jkh})_{|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} (U^i\, V_{jkh}).$$

Above equation means the decomposition tensor $(U^i V_{jkh})$ behaves as nth-recurrent. Also, we can use same technique for showing the decomposition tensors $(U_k V^i_{jh})$ and $(U^i_j V_{kh})$ are nth-recurrent if and only if the projection on indicatrix for them behave as nth-recurrent. We consider the decomposition of curvature tensor field $R^i_{jkh}$ in the following form (3.4), where $V^i_j$ is non-zero tensor field, $\emptyset_k$ and $\psi_h$ are covariant constant. Taking h-covariant derivative of (3.4) with respect to $x^m$ to nth order, we get

$$R^i_{jkh|m_1|m_2|m_3|\ldots|m_n} = V^i_{j|m_1|m_2|m_3|\ldots|m_n} \emptyset_k \psi_h \ .$$

By using the condition (2.16) in above equation and in view of (3.4), we get

(3.27) $V^i_{j|m_1|m_2|m_3|\ldots|m_n} \emptyset_k \psi_h = \lambda_{m_1 m_2 m_3 \ldots m_n} V^i_j \emptyset_k \psi_h + \mu_{m_1 m_2 m_3 \ldots m_n} (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$.

Since $\emptyset_k$ and $\psi_h$ are independent of $x^m$ to nth order, equation (3.27) can be written as

(3.28) $(V^i_j \emptyset_k \psi_h)_{|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} (V^i_j \emptyset_k \psi_h) + \mu_{m_1 m_2 m_3 \ldots m_n} (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$

In conclusion the proof of theorem is completed, we can determine,

**Theorem 3.4.** In $GR^h$-$n^{th}RF$, under the decomposition (3.4) and if $\emptyset_k$ and $\psi_h$ are covariant constant, then the decomposition tensor $(V^i_j \emptyset_k \psi_h)$ satisfies the generalized nth-recurrence property.

In view of equation (3.27), and where $\alpha^{kh}_{m_1 m_2 m_3 \ldots m_n} = \mu_{m_1 m_2 m_3 \ldots m_n} / \emptyset_k \psi_h$, we get

(3.29) $V^i_{j|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} V^i_j + \alpha^{kh}_{m_1 m_2 m_3 \ldots m_n} (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$.

Above equation can be writer as

(3.30) $V^i_{j|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} V^i_j + (\theta^i_{m_1 m_2 m_3 \ldots m_n j} - \theta^i_{m_1 m_2 m_3 \ldots m_n j})$,

where $\theta^i_{m_1 m_2 m_3 \ldots m_n j} = \alpha^{kh}_{m_1 m_2 m_3 \ldots m_n} \delta^i_h\, g_{jk}$ .

From (3.30), we get

(3.31) $V^i_{j|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} V^i_j$ .

Contracting the indices $i$ and $j$ in (3.28) using (2.7b), we get

(3.32) $(V \emptyset_k \psi_h)_{|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} (V \emptyset_k \psi_h) + \mu_{m_1 m_2 m_3 \ldots m_n} (g_{hk} - g_{kh})$ ,

where $V^i_i = V$

Since $g_{hk}$ is symmetric, we get

(3.33) $(V \emptyset_k \psi_h)_{|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} (V \emptyset_k \psi_h)$

Since $\emptyset_k$ and $\psi_h$ are independent of $x^m$ to nth order, above equation can be written:

(3.34) $V_{|m_1|m_2|m_3|\ldots|m_n} = \lambda_{m_1 m_2 m_3 \ldots m_n} V$

In conclusion the proof of theorem is completed, we can determine the corollary,

**Corollary 3.3.** Under the decomposition (3.4), if $\emptyset_k$ and $\psi_h$ are covariant constant, then the behavior of decomposition tensors $V^i_j$ , $V \emptyset_k \psi_h$ and $V$ are nth-recurrent.

## IV. CONCLUSIONS

In this research, we have conducted a comprehensive study of generalized recurrent tensor fields of R-nth order in Finsler manifolds. By employing advanced tensor calculus and differential geometry techniques, we have successfully achieved the following:

- **Decomposition theorems:** We have derived a set of novel decomposition theorems that characterize the conditions under which a generalized recurrent tensor field can be expressed as a sum of simpler tensor fields. These theorems provide a deeper understanding of the structure and properties of these tensor fields.
- **Geometric insights:** Our analysis has revealed new insights into the geometric properties of Finsler manifolds. Specifically, we have shown how the decomposition of generalized recurrent tensor fields can be used to explore the curvature and torsion of these manifolds.
- **Potential applications:** The results obtained in this study have the potential to find applications in various fields, including physics and engineering. For instance, our findings could be useful in the development of new theories of gravity or in the study of materials with complex geometric structures.

## V. RECOMMENDATIONS

Based on the findings of this research, we recommend the following avenues for future investigation:

- **Extension to other geometric structures:** It would be interesting to explore the decomposition of generalized recurrent tensor fields in other geometric structures, such as Finslerian-Lagrangian geometries or generalized Finsler geometries.
- **Applications to specific physical theories:** The results of this study could be applied to specific physical theories, such as general relativity or string theory.

- **Numerical simulations:** Numerical simulations could be used to visualize and analyze the behavior of generalized recurrent tensor fields in various scenarios.
- **Development of new invariants:** The decomposition theorems obtained in this study could be used to develop new invariants for characterizing the geometry of Finsler manifolds.

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