A New Extension of Extended Beta, Hypergeometric and confluent functions by using the product of two Wright Functions and its Applications

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Abstract—The generalizations of hypergeometric and confluent hypergeometric functions, as well as gamma and beta functions, are the subject of numerous studies. The product of two Wright functions is used in this paper to define a new type of generalized beta function. Confluent hypergeometric functions and new types of generalized Gauss functions are obtained with the aid of the generalized beta function. Additionally, certain characteristics of these functions are established, including transform formulas, Mellin transforms, derivative formulas, integral representations, and summation formulas.

Keywords — : Integral representations, summation formulas, transformation formulas, beta function, Wright function, Gauss hypergeometric function, and confluent hypergeometric functions.

I. Introduction

Some background information was provided in this section, which is necessary for the remainder of the paper. Next, the Chaudhry et al. defined confluent hypergeometric, Gauss hypergeometric, and generalized beta functions were discussed. The Wright function was utilized by Enes et al. to define the confluent, Gauss, and generalized beta hypergeometric functions respectively. Many researchers (see, for instance, ([1-24] and [26-28]) and the references therein) defined new generalizations of these functions that were motivated by the work of Chaudhry et al.

Definition of the Gamma function $\Gamma(z)$: The definite integral defines the function that Euler developed [25, 29] in order to expand the factorials to values between the integers

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \quad , \quad Re(z) > 0. \tag{1.1}$$

Beta function of Euler $B(x,y)$ (see [23, 25, 29]) is defined by:

$$B(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt, \quad (Re(x) > 0, Re(y) > 0). \tag{1.2}$$

The defined Gauss hypergeometric function and confluent hypergeometric function (see [29]) as

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (|z|<1), \tag{1.3}$$

$$(\ a,b,c \in \mathbb{C} \ \ and \ \ c \neq 0\ , -1,-2,-3,\dots),$$

where $(\delta)_n$ $(\delta \in \mathbb{C})$ is the Pochhammer symbol (see [29]) is defined by

$$(\delta)_n = \frac{\Gamma(\delta+n)}{\Gamma(\delta)}\ . \tag{1.4}$$

As stated in [25], the confluent hypergeometric function is given by

$${}_1\Phi_1(\mathrm{b};\mathrm{c};\mathrm{z}) = \sum_{\mathrm{n}=0}^{\infty} \frac{(\mathrm{b})_\mathrm{n}}{(\mathrm{c})_\mathrm{n}} \frac{\mathrm{z}^\mathrm{n}}{\mathrm{n}!}\,, \quad (|\mathrm{z}|<1), \tag{1.5}$$

$$(\ b,c \in \mathbb{C} \ \ and \ \ c \neq 0\ , -1,-2,-3,\dots).$$

The extended gamma function for $Re(x) > 0$ was provided by Chaudhry and Zubair [9] in 1994.

$$\Gamma_P(x) = \int_0^\infty t^{x-1}\, exp\left(-t - \frac{p}{t}\right) dt. \tag{1.6}$$

1997, Chaudhry et al. [10] gave the extended beta function for $Re(x) > 0,\ Re(y) > 0, Re(p) > 0$ as follows:

$$B^p(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\, exp\left(-\frac{p}{t(1-t)}\right) dt. \tag{1.7}$$

Using the newly extended beta function $B^p(\delta_1,\delta_2)$, Chaudhary et al. [12] introduced an extended hypergeometric and confluent hypergeometric functions in 2004. These functions are defined as:

$$F^p(a,b,c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B^p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \quad (1.8)$$

$$(p \geq 0, \ |z| < 1, \ Re(c) > Re(b) > 0),$$

and

$$\Phi^p(b;c;z) = \sum_{n=0}^{\infty} \frac{B^p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!},$$
$$(p \geq 0, \ Re(c) > Re(b) > 0). \quad (1.9)$$

The new confluent hypergeometric functions, extended Gauss functions, generalized gamma functions, and beta functions were provided by Enes et al. [15] in 2022:

$${}^{\Psi}\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1} \ {}_0\Psi_1\left(\alpha,\beta;-t-\frac{p}{t}\right) dt, \quad (1.10)$$

$$Re(x) > 0, \ Re(p) > 0,$$

$${}^{\Psi}B_p^{(\alpha,\beta)}(x,y) =$$
$$\int_0^1 t^{x-1}(1-t)^{y-1} \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) dt, \quad (1.11)$$

$$Re(x) > 0, \ Re(y) > 0, \ Re(p) > 0,$$

where ${}_0\Psi_1(\cdot)$ is the Wright function which defined in [16] as:

$${}_0\Psi_1(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!},$$
$$where \ \alpha,\beta \in \mathbb{C} \ and \ Re(\alpha) > -1, \quad (1.12)$$

$${}^{\Psi}F_p^{(\alpha,\beta)}(a,b,c;z) =$$
$$\sum_{n=0}^{\infty}(a)_n \frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \quad (1.13)$$

$Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$,

$${}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \quad (1.14)$$

$Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$.

II. An additional beta function extension

In this paper, we use the product of two Wright functions to define new generalizations beta function, which defined by (1.12):

Definition 1. The new generalized beta functions is defined by

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y) =$$
$$\int_0^1 t^{x-1}(1-t)^{y-1} \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) \ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dt, \quad (2.1)$$

$$Re(p) > 0, \quad Re(q) > 0, Re(x) > 0, Re(y) > 0,$$
$$\alpha,\beta \in \mathbb{R}_0^+.$$

The new beta function generalizations are referred to as Ψ -beta functions.

Theorem 2.1.

Let $Re(s) > 0, Re(x+r) > 0, Re(y+s) > 0, Re(p) > 0, Re(q) >, Re(\alpha) > -1$. Then,

$$\mathfrak{M}\left\{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\} =$$

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+r,y+s) \ {}^{\Psi}\Gamma^{(\alpha,\beta)}(r) \ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s). \quad (2.2)$$

Proof: Applying the Mellin transform on (2.1), we have

$$\mathfrak{M}\left\{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\} = \int_0^{\infty}\int_0^{\infty} p^{r-1} \ q^{s-1}$$

$$\times \int_0^1 t^{x-1}(1-t)^{y-1} \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) \ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dp\, dq\, dt,$$

$$\mathfrak{M}\left\{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\} = \int_0^1 t^{x-1}(1-t)^{y-1}$$

$$\left\{\int_0^{\infty} p^{r-1} \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) dp\right\}$$

$$\left\{\int_0^{\infty} q^{s-1} \ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dq\right\} dt, \quad (2.3)$$

substituting $u = \frac{p}{t}$ and $v = \frac{q}{(1-t)}$ in (2.3), we have

$$\mathfrak{M}\left\{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\} = \int_0^1 t^{x+r-1}(1-t)^{y+s-1}$$

$$\left\{\int_0^{\infty} u^{r-1} \ {}_0\Psi_1(\alpha,\beta;-u)\, du\right\}\left\{\int_0^{\infty} v^{s-1} \ {}_0\Psi_1(\alpha,\beta;-v) dv\right\}, \quad (2.4)$$

by applying the definition of ${}^{\Psi}\Gamma_p^{(\alpha,\beta)}(\cdot)$ to (2.4) (see [15]), we get the following desired result.

$$\mathfrak{M}\left\{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y); p \longrightarrow r, q \longrightarrow s\right\} = {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+r,y+s)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s).$$

Corollary 2.1. The following is the inverse Mellin transform of the given equation:

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)} = \frac{1}{2\pi i}\int_{\delta_1-\infty i}^{\delta_1+\infty i}\int_{\delta_2-\infty i}^{\delta_2+\infty i} {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+r,y+s)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s) p^{-r}q^{-s} dr\, ds. \qquad (2.5)$$

Theorem 2.2. Integral representations of the following kinds are valid:

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \times {}_0\Psi_1(\alpha,\beta;-p\sec^2(\theta))\ {}_0\Psi_1(\alpha,\beta;-q\csc^2(\theta))d\theta, \qquad (2.6)$$

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\ {}_0\Psi_1\left(\alpha,\beta;-p\,\frac{(1+u)}{u}\right) {}_0\Psi_1(\alpha,\beta;-q(1+u))du\,, \qquad (2.7)$$

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y) = (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1} \times {}_0\Psi_1\left(\alpha,\beta;-p\,\frac{(c-a)}{(u-a)}\right) {}_0\Psi_1\left(\alpha,\beta;-q\,\frac{(c-a)}{(c-u)}\right) du\,, \qquad (2.8)$$

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)} = 2^{1-x-y}\int_{-1}^{1}(1+u)^{x-1}(1-u)^{y-1} \times {}_0\Psi_1\left(\alpha,\beta;\frac{-2p}{(1+u)}\right) {}_0\Psi_1\left(\alpha,\beta;\frac{-2q}{(1-u)}\right) du\,. \qquad (2.9)$$

Proof: For prove the formula (2.6), putting $t = \cos^2\theta \Rightarrow dt = -2\cos\theta\sin\theta\ d\theta$ in (2.1), we have

$$(i)\quad {}^{\Psi}B_{p,q}^{(\alpha,\beta)} = 2\int_0^{\frac{\pi}{2}} (\cos^2\theta)^{x-1}\ (1-cos^2\theta)^{y-1} \times {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{cos^2\theta}\right) {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-cos^2\theta}\right)\cos\theta\sin\theta\, d\theta,$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\ \sin^{2x-1}\theta\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{cos^2\theta}\right) {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-cos^2\theta}\right)d\theta,$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\ \sin^{2x-1}\theta\ \ {}_0\Psi_1(\alpha,\beta;-p\sec^2(\theta))\ {}_0\Psi_1(\alpha,\beta;-q\csc^2(\theta))d\theta.$$

Similarly, results (2.7), (2.8) and (2.9) can be proved by taking the transformation $t = \frac{u}{1+u}$, $t = \frac{u-a}{c-a}$ and $t = \frac{1+u}{2}$ in (2.1) respectively. Thus, the proof of **theorem 2.2** is completed.

Theorem 2.3. According to this integral representation, the beta function extension satisfies

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+1,y) + {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y+1) = {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y). \qquad (2.10)$$

Proof. If we look at the left hand side of (2.10), we have

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+1,y) + {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y+1)$$

$$= \int_0^1 \{t^{x-1}(1-t)^y + t^x(1-t)^{y-1}\}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt$$

$$= {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y).$$

Theorem 2.4. Let $Re(x) > 0$, $Re(y) < 1$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,1-y) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+n,1). \qquad (2.11)$$

Proof: From (2.1), we have

$${}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,1-y) = \int_0^1 t^{x-1}(1-t)^{-y}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt,$$

applying the theorem of generalized binomials

$$(1-t)^{-y} = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^n,\quad |t|<1\,,$$

We obtain

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,1-y) $$
$$ =\sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^{x+n-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt. $$

Using (2.1) and switching the order of summation and integration in the above equation now demonstrates the intended outcome.

Theorem 2.5. According to the following infinite summation formulas, the beta function extension is satisfied:

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y)=\sum_{n=0}^{\infty}{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+n,y+1). \qquad (2.12) $$

Proof. Changing the series representation that follows in (2.1)

$$ (1-t)^{y-1}=(1-t)^{y}\sum_{n=0}^{\infty}t^{n} \qquad (|t|<1), $$

we obtain

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,1-y)=\int_0^1(1-t)^{y}\sum_{n=0}^{\infty}t^{n+x-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt, $$

the desired outcome can be obtained by utilizing (2.1) and switching the order of integration and summation in the above equation.

Theorem 2.6. The relationship described below is accurate.

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y)(x,y)= $$
$$ \sum_{k=0}^{n}\binom{n}{k}\ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y)(x+k,y+n-k). \quad n\in\mathbb{N}_0 \quad (2.13) $$

Proof. The mathematical induction on $(n\in\mathbb{N}_0)$ is used in the following way to prove (2.13).

Equation (2.13) obviously holds for $n=0$.

When $n=0$, we get:

$$ {}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x+1,y)+{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y+1) $$
$$ =\int_0^1\{t^{x}(1-t)^{y-1} $$
$$ +\ t^{x-1}(1-t)^{y}\}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt, $$

$$ =\int_0^1 t^{x-1}(1-t)^{y-1}\{\ t $$
$$ +(-t)\}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt, $$

$$ =\int_0^1 t^{x-1}(1-t)^{y-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt={}^{\Psi}B_{p,q}^{(\alpha,\beta)}(x,y). $$

Consequently, for $n=1$, the equation (2.13) is valid.

By carrying out this procedure for every $(n\in\mathbb{N}_0)$, we eventually arrive at the required relation (2.13)

III. Novel confluent hypergeometric and generalized Gauss functions

New generalized Gauss and confluent hypergeometric functions were introduced, along with some of their properties, in this section.

Definition 2. For $Re(b)>0$, $Re(p)>0$, $Re(\alpha)>-1$, respectively, the new generalized Gauss and confluent hypergeometric functions are defined by

$$ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)= $$
$$ \sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (3.1) $$

and

$$ {}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)=\sum_{n=0}^{\infty}\frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (3.2) $$

We call ${}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)$ as Ψ -Gauss hypergeometric function and ${}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)$ as Ψ -confluent hypergeometric function.

IV. Extended hypergeometric functions represented integrally

Theorem 4.1. The integral representation of the extended hypergeometric is as follows:

$$ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b)} $$
$$ \times\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ {}_0\Psi_1 $$
$$ \left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt. \qquad (4.1) $$

Proof. By using (2.1) in (3.1), we have

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b)}$$

$$\sum_{n=0}^{\infty}(a)_n\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt\,\frac{z^n}{n!},$$

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b)}$$

$$\times\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!},$$

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b)}$$

$$\times\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt.$$

Theorem 4.2. The integral representations listed below are true:

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b)}\int_0^1 u^{b-1}(1+u)^{a-c}(1-u(1-\tau))^{-a}$$

$$\times\ {}_0\Psi_1\left(\alpha,\beta;-\frac{p(1+u)}{u}\right)\ {}_0\Psi_1(\alpha,\beta;-q(1+u))du\,, \quad (4.2)$$

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=$$

$$\frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}\frac{(\sin\theta)^{2b-1}(\cos\theta)^{2c-2b-1}}{(1-\tau\sin^2\theta)^a}$$

$$(1-z\sin^2\theta)^{-a}\ {}_0\Psi_1(\alpha,\beta;-p\sin^2\theta)\ {}_0\Psi_1(\alpha,\beta;-q\csc^2\theta)d\theta \quad (4.3)$$

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=$$

$$\frac{2}{B(b,c-b)}\int_0^{\infty}\frac{(\sinh\theta)^{2b-1}(\cosh\theta)^{2c-2b+1}}{(\cosh^2\theta-\tau\sinh^2\theta)^a}$$

$${}_0\Psi_1(\alpha,\beta;-p\coth^2\theta)\ {}_0\Psi_1(\alpha,\beta;-q\cosh^2\theta)d\theta \quad (4.4)$$

$$^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)=$$

$$\frac{2^{1+a-c}}{B(b,c-b)}\int_{-1}^1(1+u)^{b-1}(1-u)^{c-b-1}$$

$$(2-z(1+u))^{-a}\ {}_0\Psi_1\left(\alpha,\beta;-p\,\frac{2}{(1+u)}\right)\ {}_0\Psi_1\left(\alpha,\beta;-q\,\frac{2}{(1-u)}\right)du. \quad (4.5)$$

Proof. By substituting $t=\frac{u}{1+u}$, $t=\sin^2\theta$, $t=\tanh^2\theta$ and $t=\frac{1+u}{2}$ in (3.1) respectively, we get the desired representations (4.2)-(4.5).

Subsequently, we present integral representations of the extended confluent hypergeometric function.

Theorem 4.3.

$$^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)=\frac{1}{B(b,c-b)}$$

$$\times\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\exp(zt)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt, \quad (4.6)$$

and

$$^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)=\frac{\exp(z)}{B(b,c-b)}$$

$$\times\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\exp(-zt)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)dt. \quad (4.7)$$

Proof. By using definition of extended beta function (2.1) in (3.2), we have

$$^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)=\frac{1}{B(b,c-b)}$$

$$\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right)\left(\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}\right)dt, \quad (4.8)$$

using

$$\sum_{n=0}^{\infty} \frac{(zt)^n}{n!} = \exp(zt),$$

we obtain the proof of (4.6) in (4.8). By replacing $t = 1 - t$ in (4.6), we can prove (4.7).

V. Differentiation formulas for the extended hypergeometric functions

We obtain differentiation formulas for the confluent and extended hypergeometric functions in this section.

Theorem 5.1. The following equation is true:

$$\frac{d}{dz}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)\right\} = \frac{(a)_n\,(b)_n}{(c)_n}\ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a+n,b+n;c+n;z). \tag{5.1}$$

Proof. When we differentiate (3.1) in relation to z, we obtain

$$\frac{d}{dz}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)\right\} = \frac{d}{dz}\sum_{n=0}^{\infty}(a)_n \frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

$$= \sum_{n=1}^{\infty}(a)_n \frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}, \tag{5.2}$$

changing n to $n+1$ in (5.2), we have

$$\frac{d}{dz}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)\right\} = \sum_{n=1}^{\infty}(a)_{n+1} \frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{5.3}$$

since

$$B(b,c-b) = \frac{c}{b}\,B(b+1,c-b). \tag{5.4}$$

Applying (5.4) in (5.3), we get

$$\frac{d}{dz}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)\right\} = \frac{a\,b}{c}\sum_{n=1}^{\infty}(a+1)_n \frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}$$

$$= \frac{a\,b}{c}\ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a+1,b+1;c+1;z), \tag{5.5}$$

again differentiating (5.5) with respect to z, we obtain

$$\frac{d^2}{dz^2}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)\right\} = \frac{(a+1)\,(b+1)}{(c+1)}\ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a+2,b+2;c+2;z), \tag{5.6}$$

continually going up to n times, we achieve the desired outcome.

Theorem 5.2. The formula shown below is valid:

$$\frac{d^n}{dz^n}\left\{ {}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)\right\} = \frac{(b)_n}{(c)_n}\ {}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b+n;c+n;z). \tag{5.7}$$

Proof. We obtain desired result by using the similar procedure as in theorem 5.1.

VI. Mellin transform of extended hypergeometric functions.

Here, we obtain the Mellin transformation of the confluent and extended hypergeometric functions (3.1) and (3.2) through a series of calculations.

Theorem 6.1. The extended hypergeometric function possesses the following Mellin transform:

$$\mathfrak{M}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z); p \longrightarrow r\,, q \longrightarrow s\right\} = \frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s)\ B(b+r,c+s-b)}{B(b,c-b)}.$$

$$\times\ {}_2F_1(a\,,b+r;\ c+r+s;z) \tag{6.1}$$

Proof. By applying the Mellin transform to both sides of equation (4.1), we obtain:

$$\mathfrak{M}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z); p \longrightarrow r\,, q \longrightarrow s\right\} = \frac{1}{B(b,c-b)}\int_0^{\infty}\int_0^{\infty} p^{r-1}\ q^{s-1} \times \int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-zt)^{-a}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dt\,dp\,dq.$$

By changing the sequence of integration in the aforementioned equation, we obtain.

$$\mathfrak{M}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z); p \longrightarrow r\,, q \longrightarrow s\right\} = \frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}\,(1-t)^{c-b-1}\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}$$

$$\times\left\{\int_0^{\infty} p^{r-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) dp\right\} \left\{\int_0^{\infty} q^{s-1}\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dq\right\} dt. \tag{6.2}$$

In

$$\left\{\int_0^\infty p \to r \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) dp\right\}$$
$$\left\{\int_0^\infty q \to s \ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dq\right\} \qquad (6.3)$$

Putting $u = \frac{p}{t}$ and $v = \frac{q}{(1-t)}$ in (6.3) and integral, we get

$$\left\{\int_0^\infty p \to r \ {}_0\Psi_1\left(\alpha,\beta;\frac{-p}{t}\right) dp\right\}$$
$$\left\{\int_0^\infty q \to s \ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dq\right\}$$
$$= t^r (1-t)^s {}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s),$$

Utilizing the equation in (6.2), we arrive at:

$$\mathfrak{M}\left\{ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z); p \to r, q \to s\right\}$$
$$= \frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s)\ B(b+r,c+s-b)}{B(b,c-b)}$$
$$\times\ {}_2F_1(a\,,b+r;\ c+r+s;z).$$

Thermo 6.2. The following conclusion is valid:

$${}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z)$$
$$= \frac{1}{(2\pi i)^2 B(b,c-b)} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty} {}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s)$$
$$\times\ B(b+r,c+s-b)\ {}_2F_1$$
$$(a\,,b+r;\ c+r+s;z)p^{-r}\ q^{-s}\ dr\,ds. \qquad (6.4)$$

Proof. By applying the inverse Mellin transform to both sides of equation (6.1), we arrive at the desired conclusion.

Theorem 6.3. The extended confluent hypergeometric function possesses the following Mellin transform:

$$\mathfrak{M}\left\{ {}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z); p \to r, q \to s\right\}$$
$$= \frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s)\ B(b+r,c+s-b)}{B(b,c-b)}$$
$$\times\,\Phi(\,b+r;\ c+r+s;z). \qquad (6.5)$$

Proof. By carrying out comparable operations to those in the Theorem 6.1 proof, the intended outcome is achieved.

Theorem 6.4. The following result holds true;

$${}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z) = \frac{1}{(2\pi i)^2 B(b,c-b)}$$
$$\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty} {}^{\Psi}\Gamma^{(\alpha,\beta)}(r)\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s) B(+r,c+s-b)$$
$$\times\,\Phi(\,b+r;\ c+r+s;z)p^{-r}\ q^{-s}\ dr\,ds\,,$$
$$(\gamma_1,\gamma_1 > 0). \qquad (6.6)$$

Proof. By performing the inverse Mellin transform on both sides on (6.3), we obtain the necessary outcome.

VII. Summation and transformation formulas

Here, we derive the transformation and summation formulas for the confluent and extended hypergeometric functions, respectively:

Theorem 7.1. This transformation is true for extended hypergeometric function for $p,q,\alpha,\beta > 0$:

$${}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;z) = (1-z)^{-a}\ {}^{\Psi}F_{p,q}^{(\alpha,\beta)}\left(a,b,c;\frac{z}{1-z}\right), \quad (7.1)$$

where $|\arg(1-z)| < \pi$.

Proof. Replacing z by $(1-z)$ in (4.1), we get the desired result.

Theorem 7.2. This transformation is true for extended hypergeometric function for $p,q,\alpha,\beta > 0$:

$${}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(b;c;z) = \exp(z)\ \ {}^{\Psi}\Phi_{p,q}^{(\alpha,\beta)}(c-b;c;-z), \qquad (7.2)$$

where $|\arg(1-z)| < \pi$.

Proof. We can easily determine the desired outcome from (4.6) and (4.7).

Theorem 7.3. The summation formula shown below is true:

$${}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b;c;1) = \frac{{}^{\Psi}B_{p,q}^{(\alpha,\beta)}(b,c-a-b)}{B(b,c-b)}, \qquad (7.3)$$

where $p,q \geq 0\ ,\quad \alpha,\beta > 0 \quad and \quad \Re(c-a-b) > 0.$

Proof. Taking $z = 1$ in (4.1), we have

$${}^{\Psi}F_{p,q}^{(\alpha,\beta)}(a,b,c;1) = \frac{1}{B(b,c-b)}$$
$$\times \int_0^1 t^{b-1}\,(1-t)^{c-a-b-1}\ {}_0\Psi_1$$
$$\left(\alpha,\beta;\frac{-p}{t}\right)\ {}_0\Psi_1\left(\alpha,\beta;\frac{-q}{1-t}\right) dt, \qquad (7.4)$$

The desired outcome can be obtained by solving the above equation using definition (2.1).

VIII. REFERENCE

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