# Comparison of Approximate Methods for Solving Delay Differential Equations with Initial value problems

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***Abstract*—This paper aims to study two approximate analytical methods for solving linear and nonlinear delay differential equations. Approximate approaches are shown in the Variational iteration method and the Adomian decomposition method. Through the conversion of some instances, including linear and nonlinear delay differential equations with initial values, by comparing different approximate methods. The results show that this procedure is accurate and adequate for DDE. Comparisons between the Adomian Decomposition Method and Variational Iteration Method results demonstrate the accuracy of the results obtained from both mentioned methods. The variational iteration method is efficient and convenient based on comparing the results and exact solutions.**

***Keywords***—*Adomian Decomposition Method, Variational iteration method, Delay differential equations, Initial value problem, Exact solution.*

## I. INTRODUCTION

Delay differential equations (DDEs) are considered in various studies and applications in different fields such as science, engineering, biology, and economics [1], [2]. We consider the general form of DDT to be

$$y'(t) = f(t, y(t), y(t-\tau), \ldots, y(t-\tau)),\ t > 0,\ t \in R^n \qquad \textbf{Eq. 1}$$

with initial function

$$\mathrm{y(t)} = \mathrm{y_0(t)} \qquad \textbf{Eq. 2}$$

where $\tau$ is the time delay; many numerical methods have been used to solve nonlinear DDEs, such as the Adomain Decomposition Method and the Variational Iteration Method. In this paper, these methods are briefly introduced.

## II. METHODOLOGY

This section discusses the approximate methods used in this approach to solve DDE.

### *A. Adomain Decomposition Method*

The Adomian decomposition method (ADM) was used before the eighties; it was developed by Adomian [1], [3] for solving linear or nonlinear ordinary and partial differential equations [10]. A significant problem in mathematics, physics, engineering, biology, chemistry, and other sciences has been solved using ADM [11-12]. A general form describes an ADM [3- 4], [9] as follows

$$\begin{aligned} &Lu(t) = f(u(x-a)),\ 0 < x < 1 \\ &u^{(i)}(0) = u^i,\ i = 0\ldots, n-1 \\ &u(t) = \phi(x);\ x \le 0, \end{aligned} \qquad \textbf{Eq. 3}$$

The differential operator $L$ is given $L(.) = \frac{d(.)}{dx^N}$ Considered $N$ fold integral using ADM for Eq. (3). Then, by applying, the inverse operator $L^{-1}$ we can write is as

$$L^{-1}Lu(x) = \int_0^x\int_0^x\int_0^x \ldots f(u(x, u(x), u(g(x))))dx_1 dx_2 \ldots dx_N \qquad \textbf{Eq. 4}$$

Operating with For the Eq. (4)

$$\mathrm{u(x)} = \sum_{n=0}^{n-1} \frac{c_j x_j}{\mathrm{j!}} + \mathrm{L}^{-1}(f(u(x, u(x), u(g(x))))) \qquad \textbf{Eq. 5}$$

where $c_j$ are constants that describe the boundary conditions. The ADM assumes that the solution $u(x)$ unknown function of Eq. (1) [5] can be expressed by an infinite series of the form

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \qquad \textbf{Eq. 6}$$

will be decomposed by an infinite series of the so-called Adomian polynomials [15]:

$$f\,(u(x,u(x),u(g(x))))=\sum_{n=0}^{\infty}A_n \qquad \textbf{Eq. 7}$$

where $A_n$ are Adomian polynomials that can be generated for all forms of nonlinearity as:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\Big(f\,\big(x,\sum_{j=0}^{\infty}\lambda^i u_j(x),\sum_{j=0}^{\infty}\lambda^i u_j(g(x))\big)\Big)_{\lambda=0} \qquad \textbf{Eq. 8}$$

We are substituting Eq. (6) and Eq. (7) into the Eq. (8) gives

$$\sum_{j=0}^{\infty}u(x)=\sum_{j=0}^{n-1}\frac{c_j x_j}{j!}+L^{-1}\Big(\sum_{n=0}^{\infty}A_n\Big) \qquad \textbf{Eq. 9}$$

Determine the components $\mathrm{u}_{\mathrm{n}}(\mathrm{x}), n\geq 0$ ,

$$u_0(x)=\sum_{j=0}^{n-1}\frac{c_j x_j}{j!},u_{n+1}(x)=L^{-1}(A_n),n\geq 0 \qquad \textbf{Eq. 10}$$

And

$$\lim_{n\to\infty}u_n(x)=u(x) \qquad \textbf{Eq. 11}$$

*B. Variational Iteration Method*

In this section, the current approach uses the variational iteration Method (VIM), which Ji Huan He first presented in 1990 [2- 3]. This technique has been applied and shown to effectively and accurately solve a large class of nonlinear problems with approximations that quickly converge to exact solutions. They have successfully used it in an ordinary differential equation [4] and a delay differential equation. To illustrate the VIM for solving a nonlinear differential equation [18] in the following form:

$$L[u(x)]+N[u(x)]=g(x) \qquad \textbf{Eq. 12}$$

Where $L$ is a linear operator, $N$ is a nonlinear operator, and $g(x)$ is the source inhomogeneous term. Agreeing to the variational iteration method, or more precisely, He's variational iteration method [1], [6], the construct of VIM a corrective functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda[Lu_n(\tau)+N\tilde{u}_n(\tau)-g(\tau)]d\tau,\quad(n\geq 0), \qquad \textbf{Eq. 13}$$

Where $\lambda$ is a general Lagrange multiplier [2], which can be identified optimally via the variational theory. The subscript $n$ denotes the approximation $\tilde{u}_n$ is considered as a restricted $\delta\tilde{u}_n=0$, [18], [21]. Thus, determine the Lagrange multiplier that will be identified optimally via integration by parts. The successive approximations of the resolution will be readily obtained upon using the obtained Lagrange multiplier and by using any selective use [23]. Then use the initial a $u_0$ oximation .Hence, it will be the solution

$$u(x)=\lim_{n\to\infty}u_n(x). \qquad \textbf{Eq. 14}$$

## III. Numerical Experimental

The main objective is to illustrate the solutions of four examples of nonlinear DDEs using ADM and VIM and compare them with the exact solution.

**Example 1** Consider the first-order of nonlinear Delay differential equations [14]

$$y'=1-2y^2\Big(\frac{x}{2}\Big);\qquad y(0)=0,\ x\in[0,1]. \qquad \textbf{Eq. 15}$$

The exact solution is

$$\mathrm{y}=\sin\,(\mathrm{x}). \qquad \textbf{Eq. 16}$$

To solve example 1, we start the ADM application, then using VIM, we get:

**Table 1. Comparing the absolute error of the ADM and VIM solution with the exact solution in example (1)**

| x | Exact solution | N=4 | | N=8 | |
|---|---|---|---|---|---|
| | | Absolute error ADM | Absolute error VIM | Absolute error ADM | Absolute error VIM |
| 0 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| 0.2 | 0.1986693307 | $1.4104\times10^{-12}$ | $6.60028\times10^{-14}$ | $2.77556\times10^{-17}$ | $2.77556\times10^{-17}$ |
| 0.4 | 0.3894183423 | $7.21349\times10^{-10}$ | $3.36373\times10^{-11}$ | 0.000000000 | 0.000000000 |
| 0.6 | 0.5646424733 | $2.76807\times10^{-8}$ | $1.282367\times10^{-9}$ | 0.000000000 | $1.11022\times10^{-16}$ |
| 0.8 | 0.7173560908 | $3.67725\times10^{-7}$ | $1.68798\times10^{-8}$ | $1.11022\times10^{-16}$ | $1.11022\times10^{-16}$ |
| 1 | 0.8414709848 | $2.73084\times10^{-6}$ | $1.23875\times10^{-7}$ | $1.11022\times10^{-16}$ | 0.000000000 |

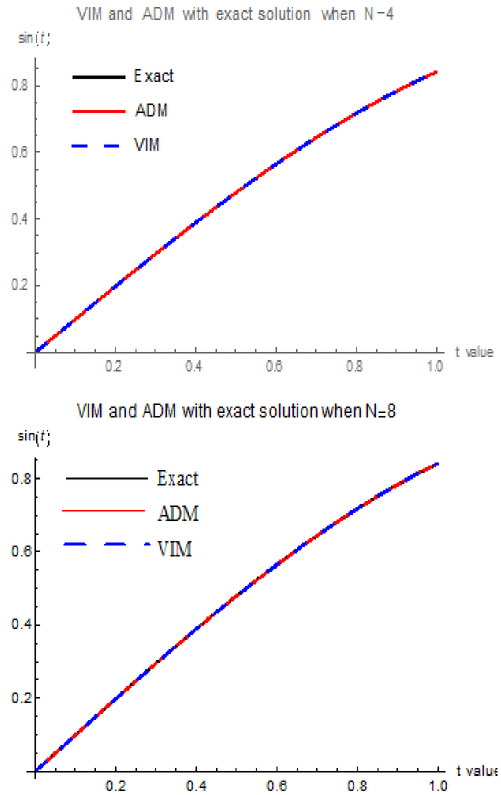


**Figure 1. Comparison between ADM and VIM shows absolute errors by using 4 and 8 terms for example (1)**

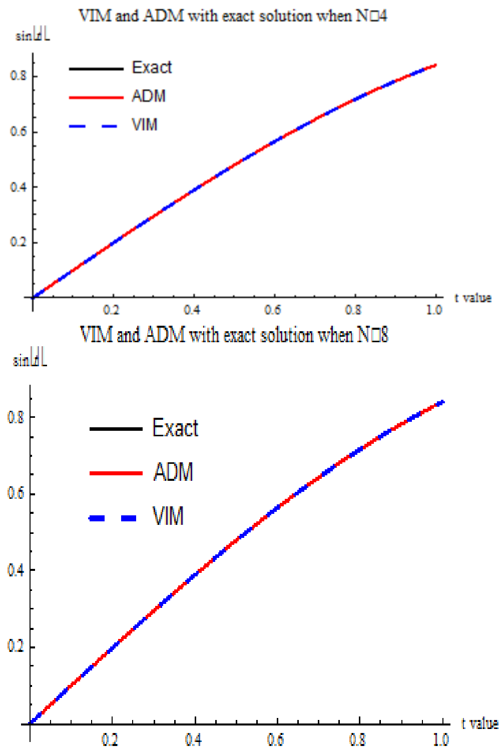


**Figure 2. Comparison between ADM and VIM with the exact solution using 4 and 8 terms for example (2)**

**Example 2** Consider the following nonlinear differential equation for the delay [14]

$$y''' = -1 + 2(y(x/2))^2;\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.;\ \ x \in [0,1].$$
$$y = \sin(x).$$

**Eq. 17**

The results obtained from solving Example 2, using both the ADM and VIM, were as follows:

**Table 2. Comparison of absolute error for ADM solution and VIM solution for example (2)**

| x | Exact solution | N=4 | | N=8 | |
|---|---|---|---|---|---|
| | | Absolute error ADM | Absolute error VIM | Absolut e error ADM | Absolute error VIM |
| 0 | 0.000000000 | 0.00000000 | 0.000000000 | 0.0000000( | 0.000000000 |
| 0.2 | 0.198669330 | 0.00000000 0 | $2.77556 \times 10^{-17}$ | $2.77556 \times 10^{-17}$ | $2.77556 \times 10^{-17}$ |
| 0.4 | 0.389418342 | $-4.996 \times 10^{-16}$ | $6.10623 \times 10^{-16}$ | 0.0000000( | 0.0000000000 |
| 0.6 | 0.564642473 | $-1.02918 \times 10^{-13}$ | $1.18572 \times 10^{-13}$ | 0.0000000( | $1.11022 \times 10^{-16}$ |
| 0.8 | 0.717356090 | $-4.32454 \times 10^{-12}$ | $4.94915 \times 10^{-12}$ | $1.11022 \times 10^{-16}$ | $1.11022 \times 10^{-16}$ |
| 1 | 0.841470984 | $-7.84507 \times 10^{-11}$ | $8.90719 \times 10^{-11}$ | $1.11022 \times 10^{-16}$ | 0.00000000 00 |

**Example 3** Consider the following delay differential equation [8]

$$y' = (1/2)e^{(x/2)}\, y(x/2) + (1/2)y(x);$$
$$y(0) = 1.\text{in } [0,1].\ y = \exp(x).$$

**Eq. 18**

The following are the results of solving Example 3 using both ADM and VIM:

**Table 3. Absolute errors with six terms, for example (3) by ADM and ADM**

| x | Exact solution | Absolute Error | |
|---|---|---|---|
| | | ADM | ADM |
| 0 | 1.0000000000000 | $2.22045 \times 10^{-16}$ | 0.210433200000 00×10-15 |
| 0.2 | 1.22140275816017 | 000000000000000 | 0.394269850063 8×10 -15 |
| 0.4 | 1.49182469764127 | $1.33227135 \times 10^{-15}$ | $0.000013334922 \times 10^{-14}$ |
| 0.6 | 1.82211880039051 | 000000000000000 | $0.00010717698 \times 10^{-11}$ |
| 0.8 | 2.22554092849247 | $4.44089352 \times 10^{-16}$ | $0.00047870163 \times 10^{-13}$ |
| 1 | 2.71828182845905 | $2.22045653 5 \times 10^{-15}$ | $0.001550636934 \times 10^{-12}$ |

**Example 4** Consider the linear delay differential equation of third-order [8]

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$$y''' = -y(x) - y(x - 0.3) + e^{(-x+0.3)};$$
$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 1. \text{ in } [0,1]. \quad \textbf{Eq. 19}$$

The exact solution is $y(x) = e^x$.
As a result of solving example 4 of a third-order DDE using the approximate solution, ADM, and VIM, we get the following results:

**Table 4: Absolute errors with six terms for Example 4 by ADM and VIM**

| x | Exact solution | Absolute Error | |
|---|---|---|---|
| | | ADM | ADM |
| 0 | 1.0000000000000 | 0.3 103244533 $^{-15}$ | 0.0000000000000 |
| 0.2 | 1.22140275816017 | 0.39426630876 | 0.0000000000000 |
| 0.4 | 1.49182469764127 | 0.3075607048985 | 0.30756087427 |
| 0.6 | 1.82211880039051 | 0.0150697663345$^{-15}$ | 0.015069760648$^{-16}$ |
| 0.8 | 2.22554092849247 | 0.015060505645$^{-14}$ | 0.0000000000000 |
| 1 | 2.71828182845905 | 0.016364763345$^{-13}$ | 0.0163647874$^{-15}$ |

## IV. Results and Discussion

Organize this section based on stated objectives, a chronological timeline, case groups, trial arrangements, or any logical arrangement that makes sense.

### *A. Results*

The main objective is to conduct a comparative study between VIM and ADM. The two methods are practical and efficient procedures that provide approximations with higher accuracy and solutions in the closed form if they exist. As mentioned earlier, eight iterations in VIM are equivalent to eight iterations in ADM. In our work, we use Mathematica 10 to compute the results of ADM and VIM. This study illustrates how VIM and ADM can be used to obtain approximate analytical solutions to a nonlinear problem arising in delay equations. The comparison between the solution of the fourth iteration of VIM and the four terms of ADM is shown in Figure (1) for N= 4 and N = 8, and excellent agreement is observed. Solutions for DDE were obtained using the methods VIM and ADM. All numerical results obtained with some terms of VIM and ADM show excellent agreement with the exact solutions. Comparing our results with the two previous methods shows that the considered methods are the most reliable, powerful, and promising. The accuracy of the methods VIM and ADM varies concerning the time t, which does not provide better approximations when the time becomes more significant than thirty seconds; in other words, the methods VIM and ADM do not give better approximations when the time increases. The method VIM has been successfully applied to find the solution to problems - we compared the results obtained with VIM with the results obtained with ADM. Table 1 - Table 4 shows the apparent superiority of VIM over ADM for this type of DDEs.

### *B. Discussions*

The VIM is better than the ADM and more accurate than the ADM. The relative error with the VIM was consistently small compared to the ADM method for all three problems in this study. We can come to an important conclusion here. VIM provides several successive approximations using the iteration of the correction function. VIM requires the evaluation of the Lagrangian multiplier $\lambda$, while ADM involves the review of the Adomian polynomials, which usually need lengthy algebraic calculations.
Interestingly, unlike the successive approximations of VIM, ADM provides the solution in even components, which are added to obtain the series solution. VIM reduces the amount of computation since no Adomian polynomials are required; therefore, the iteration is direct. However, ADM involves using Adomian polynomials for nonlinear terms, which requires more work. We solved DDE using ADM and VIM. Based on the examples, we found that ADM is a more accurate method, as the results match VIM. ADM and VIM were, therefore, remarkably effective in solving DDE VIM plays an excellent role in mathematics and engineering.

## V. Conclusion

The main insights of this study of ADM are based on the decomposition of a nonlinear DDE into a series of forms. We obtain each series term from a polynomial generated by a power series expansion of a nonlinear DDE. The technique presents a difficulty in computing the polynomial but is simple in the formulation. We have shown that this technique is an excellent tool for solving various DDEs.

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