Finsler Space of Fourth Order on Some Properties of R^h-Generalized Recurrent

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Received: 8 February 2023
Revised: 20 September 2023
Accepted: 22 September 2023

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https://doi.org/10.20428/jst.v28i2.2034

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***Abstract*— In the present paper, a Finsler space F_n whose Cartan's fourth curvature tensor R^i_{jkh} satisfies $R^i_{jkh|\ell|m|n|s} = c_{\ell mns} R^i_{jkh} + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where $|l|m|n|s$ is h-covariant derivative of fourth order (Cartan's third kind covariant differential operator), with respect to x^l , x^m , x^n and x^s successively, $c_{\ell mns}$ and $d_{\ell mns}$ are non-zero covariant vector field and covariant tensor field of third order, respectively, is introduced and such space is called as R^h-generalized fourecurrent Finsler space and we denote them briefly by R^h-G-FR F_n , we obtained some generalized fourecurrent space. Also we introduced Ricci generalized fourecurrent space.**

Keywords— Finsler space, Cartan's fourth curvature tensor R_jkh^i, Ricci generalized tensor, generalized fourecurrent tensors.

I. Introduction

A 3-dimensional Riemannian space of recurrent was introduced and studied by H. Rund[16].The generalized curvature tensors in recurrent Finsler space used the sense of Berwald and Cartan curvature tensor discussed by AL-Qashbari A.M.A. and other ([1], [3], [4], [5], [6], [7], [8] and [9]). Some properties for Weyl's projective curvature tensor studied by AL-Qashbari A.M.A. [2]. The generalized birecurrent, trirecurrent Finsler space and higher order recurrent are studied in ([13], [14], [16], [17], [19] and [20]). H.S. Ruse [17] introduced and studied a three dimensional space as space of recurrent curvature. The recurrent of an n-dimensional space was extended to Finsler space ([11], [15], [18]) for the first time. Due to different connections of Finsler space, the recurrence of different curvature tensors have been discussed by R.S. Mishra and H.D. Pande [23] and P.N. Pandey [21]. S. Dikshit [10] discussed a Finsler space in which Cartan's third curvature tensor R^i_{jkh} is birecurrent. F.Y.A. Qasem [6] discussed a Finsler space in which Cartan's third curvature tensor R^i_{jkh} is is birecurrent. F.Y.A. Qasem [6] discussed a Finsler space in which Cartan's third curvature tensor $Rjkhi$ is birecurrent of the first and second kind. F.Y.A. Qasem and A. A. M. Saleem [12] discussed a Finsler space h-curvature tensor U$jkhi$. A.M.A. Al-Qashbari [1] introduced the Rh-recurrent space, the Rh-recurrent space is characterized by

$Rjkh|\ell i = \lambda\ell Rjkhi + \mu\ell(\delta kigjh - \delta higjk)$, $Rjkhi \neq 0$,

where $\lambda\ell$ is non-zero covariant vector field known by the recurrence vector field.

W.H.A. Hadi [15] discussed the Rh-birecurrent space. Thus, the Rh-birecurrent space is characterized by

$Rjkh|\ell|mi = a\ell m\, Rjkhi + b\ell m(\delta kigjh - \delta higjk)$, $Rjkhi \neq 0$,

where $a\ell m$ is non-zero covariant tensor field of second order known by the birecurrence tensor field.

The metric tensor gij and the associate metric tensor gij are covariant constant with respect to h-covariant derivative, i.e.

(1.3) a) $g_{ij|k} = 0$ and b) $g^{ij}_{|k} = 0$.

(1.4) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$.

The covariant derivative of the vectors y^i and y_i , vanish identically, i.e.

(1.5) a) $y^i_{|k} = 0$ and b) $y_{i|k} = 0$.

(1.6) a) $y_i\, y^i = F^2$ and b) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$.

The vectors y_i and δ^i_k also satisfy the following relations

(1.7) a) $\delta^i_k\, y^k = y^i$ and b) $\delta^i_k\, y_i = y_k$

(1.8) a) $\delta^i_j\, g^{jk} = g^{ik}$ and b) $\delta^i_k\, \delta^k_h = \delta^i_h$

(1.9) a) $\delta^i_k\, g_{ji} = g_{jk}$ and b) $g_{jh}\, y^j = y_h$

Using Euler,s theorem on homogeneous properties, this tensor satisfies the following identities

(1.10) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$

and b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$.

(1.11) $C_{ijk} = g_{hj}\, C^h_{ik}$

The associate curvature tensor R_{ijkh} of the curvature tensor R^i_{jkh} is given by

(1.12) a) $R_{ijkh} = g_{rj}\, R^r_{ikh}$

and b) $R_{jrkh}\, g^{ir} = R^i_{jkh}$.

The R-Ricci tensor R_{jk} , the curvature scalar R and the deviation tensor H_j

(1.13) a) $R^i_{jki} = R_{jk}$, b) $R_{jk}\, g^{jk} = R$

and c) $H^i_{ji} = H_j$.

The curvature tensor R^i_{jkh} and curvature tensor H^i_{jkh}, satisfies the relations

(1.14) a) $R^i_{jkh}\, y^j = H^i_{kh}$, b) $H^i_{jkh} = \partial_j H^i_{kh}$

and c) $H^i_{jki} = H_{jk}$.

The scalar curvature H are given by

(1.15) $H^i_i = (1-n)\, H$.

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they are called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)–torsion tensor, respectively and are defined as follow [16]:

(1.16) a) $H^i_{jkh} \coloneqq \partial_j\, G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k \; - h/k$

and

b) $H^i_{kh} = \partial_h\, G^i_k \; + \; G^r_k\, C^i_{rh} \; - \; h/k$.

They are also related by [19]

(1.17) a) $H^i_{jkh}\, y^j = H^i_{kh}$, b) $g_{rp}\, H^i_{jkh} = H_{jpkh}$

and c) $H^i_{jk} = \dot{\partial}_j\, H^i_k$.

The process of h-covariant differentiation, with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to [16]

(1.18) $\dot{\partial}_j\left(X^i_{|k}\right) - \left(\dot{\partial}_j\, X^i\right)_{|k} = X^r\left(\dot{\partial}_j\, \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$.

In view of Euler's theorem on homogeneous functions we have:

(1.19) a) $H^i_{jk}\, y^j = -\, H^i_{kj}\, y^j = H^i_k$

and b) $g_{ip}\, H^i_{jk} = H_{jpk}$.

Tensors P^i_{hk} and P^i_k are called P-Ricci tensor and the curvature scalar, respectively defined by

(1.20) a) $P^i_{hk}\, y^h = P^i_k$

and b) $R^i_{jkh} = K^i_{jkh} + C^i_{jm}\, H^m_{kh}$.

Also, the curvature tensor R^i_{jkh} and its associate tensor R_{ijhk} satisfies the following identities known as Bianchi identities

(1.21) $R^i_{hjk} + R^i_{jkh} + R^i_{khj} - \left(C^i_{hr}H^r_{jk} + C^i_{jr}H^r_{kh} + C^i_{kr}H^r_{jh}\right) = 0$

An n-dimensional Finsler space, Fig.(1.1), equipped with the metric function f satisfies the requisite conditions [13]. Consider the components of the corresponding metric tensor g_{ij} , Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk} (the indices i, j, k , ... assume positive integral values from 1 to n). These are symmetric in their lower indices.

Fig. (1.1):
Finsler Space as a Locally Minkowskian Space

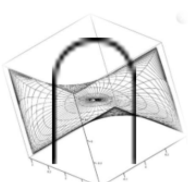

Fig. (1.2):
Metric tensor g_{ij}

Fig. (1.3) :
Metric tensor g^{ij}

II. On Necessary and Sufficient Condition of Generalized R^h-Fourecurrent

Let us consider a Finsler space F_n in which Cartan's third curvature tensor R^i_{jkh} satisfied the following generalized recurrence condition

(2.1) $R^i_{jkh|\ell} = \lambda_\ell\, R^i_{jkh} + \mu_\ell(\, \delta^i_k g_{jh} - \delta^i_h g_{jk}\,), \quad R^i_{jkh} \neq 0$

where $|l$ is h-covariant derivative of first order (Cartan's third kind covariant differential operator) with respect to x^l and λ_ℓ , μ_ℓ are non-null covariant vectors field and such space is called it generalized R^h-recurrent space.

Let us consider a Finsler space F_n for which Cartan's third curvature tensor R^i_{jkh} satisfied the following generalized birecurrence condition

(2.2) $R^i_{jkh|\ell|m} = a_{\ell m}\, R^i_{jkh} + b_{\ell m}(\, \delta^i_k g_{jh} - \delta^i_h g_{jk}\,), \quad R^i_{jkh} \neq 0$

where $|l|m$ is h-covariant derivative of second order (Cartan's third kind covariant differential operator) with respect to x^l and x^m, successively, α_{lm} and β_{lm} are non-null covariant vectors field and such space is called it generalized R^h-birecurrent space.

Taking h-covariant derivative of (2.2) with respect to x^n and using (1.5a), we get

(2.3) $R^i_{jkh|\ell|m|n} = c_{\ell mn} R^i_{jkh} + \; d_{\ell mn}\left(\, \delta^i_k g_{jh} - \delta^i_h g_{jk}\,\right)$,

$R^i_{jkh} \neq 0$

where $|\ell|m|n$ is h-covariant derivative of third order with respect to x^ℓ , x^m and x^n successfully, $c_{\ell mn} = a_{\ell|mn} + a_{\ell m}\, \lambda_n$ and $d_{\ell mn} = a_{\ell m}\mu_{nm} \;\; iesscalar\, s + b_{\ell|m|n}$ are non-zero covariant tensors fields of third order, called recurrence tensors field.

Taking h-covariant derivative of (2.3) with respect to x^s and using (1.5a), we get

(2.4) $R^i_{jkh|\ell|m|n|s} = c_{\ell mns} R^i_{jkh} + \; d_{\ell mns}\left(\, \delta^i_k g_{jh} - \delta^i_h g_{jk}\,\right)$,

$R^i_{jkh} \neq 0$

where $|\ell|m|n|s$ is h-covariant derivative of four order with respect to x^ℓ , x^m , x^n and x^s successfully, $c_{\ell mns} = a_{\ell|mns} + a_{\ell ms}\lambda_n + a_{\ell m}\lambda_{ns}$ and $d_{\ell mns} = a_{\ell ms}\, \mu_{nm} \;\; iesscalar\, s + a_{\ell m}\, \mu_{nsm} \;\; iesscalar\, s + b_{\ell|m|n|s}$ are non-zero covariant tensors fields of four order, called recurrence tensors field.

The space and the tensor satisfying the condition (2.4) is called R^h-generalized fourecurrent space. We shall denote them briefly by R^h-G-FR F_n .

A. Result

Every generalized R^h-Fourecurrent space is generalized R^h-Trirecurrent space.

Transvecting the condition (2.4) by g_{ir} , using (1.3a), (1.12a) and (1.9a), we get

(2.5) $R_{jrkh|\ell|m|n|s} = c_{\ell mns} R_{jrkh} + d_{\ell mns}\left(\, g_{kr} g_{jh} - g_{hr} g_{jk}\,\right), \quad R_{jrkh} \neq 0$

https://doi.org/10.20428/jst.v28i2.2034

Conversely, the transvection of the condition (2.4) by g^{ir}, by using (1.3b), (1.12b) and (1.4), yields the condition (2.4)

Thus, we may conclude

Transvecting the condition (2.4) by y^j, using (1.14a), (1.9b) and (1.5a), we get

(2.6) $H^i_{kh|\ell|m|n|s} = c_{\ell mns} H^i_{kh} + d_{\ell mns}(\delta^i_k y_h - \delta^i_h y_k)$.

Transvecting (2.6) by y^k, using (1.17a), (1.7a), (1.6a) and (1.5a), we get

(2.7) $H^i_{h|\ell|m|n|s} = c_{\ell mns} H^i_h + d_{\ell mns}(y^i y_h - \delta^i_h F^2)$.

Thus, we may conclude

B. Theore 2.1. In R^h-G-FRF$_n$, the h-covariant derivative of fourth order for the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h given by (2.6) and (2.7), respectively.

Differentiating (2.6), partially with respect to y^j, using (1.14b) and (1.6b), we get

(2.8) $\dot\partial_j\left(H^i_{kh|\ell|m|n|s}\right) = (\dot\partial_j c_{\ell mns}) H^i_{kh} + c_{\ell mns} H^i_{jkh} + (\dot\partial_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using the commutation formula exhibited by (1.16) for $\left(H^i_{kh|\ell|m|n}\right)$ in (2.8), we get

(2.9) $\dot\partial_j(H^i_{kh|\ell|m|n})_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - H^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{ls}) - H^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - H^i_{kh|l|m|r}(\dot\partial_j \Gamma^{*r}_{ns}) - \dot\partial_r\left(H^i_{kh|\ell|m|n}\right) P^r_{js}$
$= (\dot\partial_j c_{\ell mns}) H^i_{kh} + c_{\ell mns} H^i_{jkh} + (\dot\partial_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Again, applying the commutation formula exhibited by (1.16) for $\left(H^i_{kh|\ell|m}\right)$ in (2.9), we get

(2.10) $\left\{\dot\partial_j\left(H^i_{kh|\ell|m}\right)\right\}_{|n|s} + [H^r_{kh|\ell|m}(\dot\partial_j \Gamma^{*i}_{rn}) - H^i_{rh|\ell|m}(\dot\partial_j \Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot\partial_j \Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r\left(H^i_{kh|\ell|m}\right) P^r_{jn}]_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - H^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{\ell s}) - H^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - H^i_{kh|l|m|r}(\dot\partial_j \Gamma^{*r}_{ns}) - [\{\dot\partial_r\left(H^i_{kh|\ell|m|n}\right)\}_{|s} + H^q_{kh|\ell|m|n}(\dot\partial_r \Gamma^{*i}_{qs}) - H^i_{qh|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{ks}) - H^i_{kq|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{hs}) - H^i_{kh|q|m|n}(\dot\partial_r \Gamma^{*q}_{ls}) - H^i_{kh|l|q|n}(\dot\partial_r \Gamma^{*q}_{ms}) - H^i_{kh|l|m|q}(\dot\partial_r \Gamma^{*q}_{ns}) - \dot\partial_q(H^i_{kh|\ell|m}) P^q_{rn}] P^r_{js} = (\dot\partial_j c_{\ell mns}) H^i_{kh} + c_{\ell mns} H^i_{jkh} + (\dot\partial_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Again, applying the commutation formula exhibited by (1.16) for $\left(H^i_{kh|l}\right)$ in (2.10), we get

(2.11) $(\dot\partial_j H^i_{kh|l})_{|m|n|s} + \{H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm}) - (\dot\partial_r H^i_{kh|l}) P^r_{jm}\}_{|n|s} + [H^r_{kh|\ell|m}(\dot\partial_j \Gamma^{*i}_{rn}) - H^i_{rh|\ell|m}(\dot\partial_j \Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot\partial_j \Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r\left(H^i_{kh|\ell|m}\right) P^r_{jn}]_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - H^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{\ell s}) - H^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - H^i_{kh|l|m|r}(\dot\partial_j \Gamma^{*r}_{ns}) - [\{\dot\partial_r\left(H^i_{kh|\ell|m|n}\right)\}_{|s} + H^q_{kh|\ell|m|n}(\dot\partial_r \Gamma^{*i}_{qs}) - H^i_{qh|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{ks}) - H^i_{kq|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{hs}) - H^i_{kh|q|m|n}(\dot\partial_r \Gamma^{*q}_{ls}) - H^i_{kh|l|q|n}(\dot\partial_r \Gamma^{*q}_{ms}) - H^i_{kh|l|m|q}(\dot\partial_r \Gamma^{*q}_{ns}) - \dot\partial_q(H^i_{kh|\ell|m}) P^q_{rn}] P^r_{js}$
$= (\dot\partial_j c_{\ell mns}) H^i_{kh} + c_{\ell mns} H^i_{jkh} + (\dot\partial_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Further, applying the commutation formula exhibited by (1.16) for $\left(H^i_{kh}\right)$ in (2.11), we get

(2.11) $(\dot\partial_j H^i_{kh})_{|l|m|n|s} + \{H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hl}) - (\dot\partial_r H^i_{kh}) P^r_{jl}\}_{|m|n|s} + \{H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm}) - (\dot\partial_r H^i_{kh|l}) P^r_{jm}\}_{|n|s} + [H^r_{kh|\ell|m}(\dot\partial_j \Gamma^{*i}_{rn}) - H^i_{rh|\ell|m}(\dot\partial_j \Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot\partial_j \Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r\left(H^i_{kh|\ell|m}\right) P^r_{jn}]_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - H^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{\ell s}) - H^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - H^i_{kh|l|m|r}(\dot\partial_j \Gamma^{*r}_{ns}) - [\{\dot\partial_r\left(H^i_{kh|\ell|m|n}\right)\}_{|s} + H^q_{kh|\ell|m|n}(\dot\partial_r \Gamma^{*i}_{qs}) - H^i_{qh|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{ks}) - H^i_{kq|\ell|m|n}(\dot\partial_r \Gamma^{*q}_{hs}) - H^i_{kh|q|m|n}(\dot\partial_r \Gamma^{*q}_{ls}) - H^i_{kh|l|q|n}(\dot\partial_r \Gamma^{*q}_{ms}) - H^i_{kh|l|m|q}(\dot\partial_r \Gamma^{*q}_{ns}) - \dot\partial_q(H^i_{kh|\ell|m}) P^q_{rn}] P^r_{js} = (\dot\partial_j c_{\ell mns}) H^i_{kh} + c_{\ell mns} H^i_{jkh} + (\dot\partial_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using (1.17b) in (2.11), we get

(2.12) $H^i_{jkh|l|m|n|s} + \{H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hl}) - (\dot\partial_r H^i_{kh}) P^r_{jl}\}_{|m|n|s} + \{H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm}) - (\dot\partial_r H^i_{kh|l}) P^r_{jm}\}_{|n|s} + [H^r_{kh|\ell|m}(\dot\partial_j \Gamma^{*i}_{rn}) - H^i_{rh|\ell|m}(\dot\partial_j \Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot\partial_j \Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r\left(H^i_{kh|\ell|m}\right) P^r_{jn}]_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs})$

$- H^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{\ell s}) - H^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) -$
$H^i_{kh|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\{\dot{\partial}_r(H^i_{kh|\ell|m|n})\}_{|s}$
$+ H^q_{kh|\ell|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - H^i_{qh|\ell|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) -$
$H^i_{kq|\ell|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - H^i_{kh|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls})$
$- H^i_{kh|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - H^i_{kh|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) -$
$\dot{\partial}_q(H^i_{kh|\ell|m})P^q_{rn}]\, P^r_{js} = (\dot{\partial}_j c_{\ell mns})\, H^i_{kh}$
$+ c_{\ell mns}\, H^i_{jkh} + (\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) +$
$d_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

This shows that

(2.13) $H^i_{jkh|\ell|m|n|s} = c_{\ell mns}\, H^i_{jkh} + d_{\ell mns}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$

if and only if

(2.14) $\{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) -$
$(\dot{\partial}_r H^i_{kh})P^r_{jl}\}_{|m|n|s} + \{H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm})$
$- H^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) -$
$(\dot{\partial}_r H^i_{kh|l})P^r_{jm}\}_{|n|s} + [H^r_{kh|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{rn})$
$- H^i_{rh|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) -$
$H^i_{kh|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn})$
$- \dot{\partial}_r(H^i_{kh|\ell|m})\, P^r_{jn}\,]_{|s} + H^r_{kh|\ell|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) -$
$H^i_{kr|\ell|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs})$
$- H^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{\ell s}) - H^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - H^i_{kh|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns}) -$
$[\{\dot{\partial}_r(H^i_{kh|\ell|m|n})\}_{|s}$
$+ H^q_{kh|\ell|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - H^i_{qh|\ell|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) -$
$H^i_{kq|\ell|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - H^i_{kh|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls})$
$- H^i_{kh|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - H^i_{kh|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) -$
$\dot{\partial}_q(H^i_{kh|\ell|m})P^q_{rn}]\, P^r_{js} - (\dot{\partial}_j c_{\ell mns})\, H^i_{kh}$
$- (\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) = 0$.

Thus, we may conclude

Theorem 2.2. In R^h-G-FRF_n , Berwald curvature tensor H^i_{jkh} is generalized fourecurrent tensor if and only if (2.14) hold good.

Transvecting (2.12) by g_{ip} , using (1.3a), (1.17b), (1.19b) and (1.9a), we get

(2.15) $H_{jpkh|\ell|m|n|s} + \{g_{ip}\, H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H_{rph}(\dot{\partial}_j\Gamma^{*r}_{kl}) -$
$H_{kpr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - g_{ip}\,(\dot{\partial}_r H^i_{kh})P^r_{jl}\}_{|m|n|s}$
$+ \{g_{ip}\, H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H_{rph|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - H_{kpr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) -$
$H_{kph|r}(\dot{\partial}_j\Gamma^{*r}_{lm})$
$- g_{ip}\,(\dot{\partial}_r H^i_{kh|l})\, P^r_{jm}\}_{|n|s} + [\, g_{ip}\, H^r_{kh|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) -$
$H_{rph|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H_{kpr|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{hn})$
$-H_{kph|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H_{kph|m|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) - g_{ip}\,\dot{\partial}_r(H^i_{kh|\ell|m})\, P^r_{jn}]_{|s} +$
$g_{ip}\, H^r_{kh|\ell|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs})$
$-H_{rph|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - H_{kpr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - H_{kph|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{\ell s}) -$
$H_{kph|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms})$
$-H_{kph|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\{g_{ip}\dot{\partial}_r(H^i_{kh|\ell|m|n})\}_{|s} +$
$g_{ip}\, H^q_{kh|\ell|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - H_{qph|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks})$
$-H_{kpq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - H_{kph|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) -$
$H_{kph|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - H_{kph|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns})$
$-g_{ip}\,\dot{\partial}_q(H^i_{kh|\ell|m})P^q_{rn}]\, P^r_{js} = (\dot{\partial}_j c_{\ell mns})\, H_{kph} +$
$c_{\ell mns}\, H_{jpkh}$
$+ g_{ip}\,(\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mns}(g_{kp}\, g_{jh} -$
$g_{hp} g_{jk})$.

This shows that

(2.16) $H_{jpkh|\ell|m|n|s} = c_{\ell mns}\, H_{jpkh} + d_{\ell mns}(\, g_{kp}\, g_{jh} -$
$g_{hp}\, g_{jk}\,)$

if and only if

(2.17) $\{\, g_{ip}\, H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H_{rph}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H_{kpr}(\dot{\partial}_j\Gamma^{*r}_{hl}) -$
$g_{ip}\,(\dot{\partial}_r H^i_{kh})P^r_{jl}\}_{|m|n|s}$
$+ \{\, g_{ip}\, H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H_{rph|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - H_{kpr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) -$
$H_{kph|r}(\dot{\partial}_j\Gamma^{*r}_{lm})$
$- g_{ip}\,(\dot{\partial}_r H^i_{kh|l})\, P^r_{jm}\}_{|n|s} + [\, g_{ip}\, H^r_{kh|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) -$
$H_{rph|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H_{kpr|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{hn})$
$-H_{kph|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H_{kph|m|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) - g_{ip}\,\dot{\partial}_r(H^i_{kh|\ell|m})\, P^r_{jn}]_{|s} +$
$g_{ip}\, H^r_{kh|\ell|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs})$
$-H_{rph|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - H_{kpr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - H_{kph|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{\ell s}) -$
$H_{kph|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms})$
$-H_{kph|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\{g_{ip}\dot{\partial}_r(H^i_{kh|\ell|m|n})\}_{|s} +$
$g_{ip}\, H^q_{kh|\ell|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - H_{qph|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks})$
$-H_{kpq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - H_{kph|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) -$
$H_{kph|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - H_{kph|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns})$
$- g_{ip}\,\dot{\partial}_q(H^i_{kh|\ell|m})P^q_{rn}]\, P^r_{js} - (\dot{\partial}_j c_{\ell mns})\, H_{kph} -$
$g_{ip}\,(\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) = 0$.

Thus, we may conclude

Theorem 2.3. In R^h-G-FRF_n , the associative curvature tensor H_{jpkh} of Berwald curvature tensor H^i_{jkh} is generalized fourecurrent tensor if and only if (2.16) holds good.

Contracting the indices i and h in (2.12), using (1.7b), (1.14c) and (1.13c), we get

(2.18) $H_{jk|l|m|n|s} + \{\, H^r_{ki}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H_r\,(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{il}) -$
$(\dot{\partial}_r H_k\,)P^r_{jl}\}_{|m|n|s}$
$+\{H^r_{ki|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H_{r|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{im}) - H_{k|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) -$
$(\dot{\partial}_r H_{k|l})P^r_{jm}\}_{|n|s}$
$+ [\, H^r_{ki|\ell|m}(\dot{\partial}_j\Gamma^{*i}_{rn}\,) - H_{r|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot{\partial}_j\Gamma^{*r}_{in}) -$
$H_{k|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln})$

$- \dot{\partial}_r \left(H_{k|\ell|m}\right) P_{jn}^r]_{|s} + H_{ki|\ell|m|n}^r \left(\dot{\partial}_j \Gamma_{rs}^{*i}\right) - H_{r|\ell|m|n} \left(\dot{\partial}_j \Gamma_{ks}^{*r}\right) - H_{kr|\ell|m|n}^i \left(\dot{\partial}_j \Gamma_{is}^{*r}\right)$

$- H_{k|r|m|n} \left(\dot{\partial}_j \Gamma_{\ell s}^{*r}\right) - H_{k|l|r|n} \left(\dot{\partial}_j \Gamma_{ms}^{*r}\right) - H_{k|l|m|r} \left(\dot{\partial}_j \Gamma_{ns}^{*r}\right) - [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s}$

$+ H_{ki|\ell|m|n}^q \left(\dot{\partial}_r \Gamma_{qs}^{*i}\right) - H_{q|\ell|m|n} \left(\dot{\partial}_r \Gamma_{ks}^{*q}\right) - H_{kq|\ell|m|n}^i \left(\dot{\partial}_r \Gamma_{is}^{*q}\right) - H_{k|q|l|m|n} \left(\dot{\partial}_r \Gamma_{ls}^{*q}\right)$

$- H_{k|l|q|n} \left(\dot{\partial}_r \Gamma_{ms}^{*q}\right) - H_{k|l|m|q} \left(\dot{\partial}_r \Gamma_{ns}^{*q}\right) - \dot{\partial}_q (H_{k|\ell|m}) P_{rn}^q] P_{js}^r = \left(\dot{\partial}_j c_{\ell mns}\right) H_k$

$+ c_{\ell mns} H_{jk} + \left(\dot{\partial}_j d_{\ell mns}\right) (y_k - y_h) + d_{\ell mns} (1 - n) g_{jk}$.

This shows that

(2.19) $H_{jk|l|m|n|s} = c_{\ell mns} H_{jk} + (1 - n)\, d_{\ell mns}\, g_{jk}$

If and only if

(2.20) $\{ H_{ki}^r \left(\dot{\partial}_j \Gamma_{rl}^{*i}\right) - H_r \left(\dot{\partial}_j \Gamma_{kl}^{*r}\right) - H_{kr}^i \left(\dot{\partial}_j \Gamma_{il}^{*r}\right) - (\dot{\partial}_r H_k) P_{jl}^r \}_{|m|n|s} + \{ H_{ki|l}^r \left(\dot{\partial}_j \Gamma_{rm}^{*i}\right)$

$- H_{r|l} \left(\dot{\partial}_j \Gamma_{km}^{*r}\right) - H_{kr|l}^i \left(\dot{\partial}_j \Gamma_{im}^{*r}\right) - H_{k|r} \left(\dot{\partial}_j \Gamma_{lm}^{*r}\right) - (\dot{\partial}_r H_{k|l}) P_{jm}^r \}_{|n|s} + [H_{ki|\ell|m}^r \left(\dot{\partial}_j \Gamma_{rn}^{*i}\right)$

$- H_{r|\ell|m} \left(\dot{\partial}_j \Gamma_{kn}^{*r}\right) - H_{kr|\ell|m}^i \left(\dot{\partial}_j \Gamma_{in}^{*r}\right) - H_{k|r|m} \left(\dot{\partial}_j \Gamma_{ln}^{*r}\right) - \dot{\partial}_r \left(H_{k|\ell|m}\right) P_{jn}^r]_{|s}$

$+ H_{ki|\ell|m|n}^r \left(\dot{\partial}_j \Gamma_{rs}^{*i}\right) - H_{r|\ell|m|n} \left(\dot{\partial}_j \Gamma_{ks}^{*r}\right) - H_{kr|\ell|m|n}^i \left(\dot{\partial}_j \Gamma_{is}^{*r}\right) - H_{k|r|m|n} \left(\dot{\partial}_j \Gamma_{\ell s}^{*r}\right)$

$- H_{k|l|r|n} \left(\dot{\partial}_j \Gamma_{ms}^{*r}\right) - H_{k|l|m|r} \left(\dot{\partial}_j \Gamma_{ns}^{*r}\right) - [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s} + H_{ki|\ell|m|n}^q \left(\dot{\partial}_r \Gamma_{qs}^{*i}\right)$

$- H_{q|\ell|m|n} \left(\dot{\partial}_r \Gamma_{ks}^{*q}\right) - H_{kq|\ell|m|n}^i \left(\dot{\partial}_r \Gamma_{is}^{*q}\right) - H_{k|q|l|m|n} \left(\dot{\partial}_r \Gamma_{ls}^{*q}\right) - H_{k|l|q|n} \left(\dot{\partial}_r \Gamma_{ms}^{*q}\right)$

$- H_{k|l|m|q} \left(\dot{\partial}_r \Gamma_{ns}^{*q}\right) - \dot{\partial}_q (H_{k|\ell|m}) P_{rn}^q] P_{js}^r - \left(\dot{\partial}_j c_{\ell mns}\right) H_k - \left(\dot{\partial}_j d_{\ell mns}\right) (y_k - y_h) = 0$

The equation (2.19) shows that the H–Ricci tensor H_{jk} can't vanish, because the vanishing of it would implies $d_{\ell mns} = 0$, if and only if (2.20) hold good, a contradiction.

Thus, we may conclude

Theorem 2.4. In R^h-G-FRF$_n$, the H-Ricci tensor H_{jk} can't vanish if and only if (2.20) holds good .

III. On Generalized R^h-Fourecurrent-Affinely Connected Space

In this section, we shall introduce new definition for R^h-G-FRF$_n$, whose also possess the properties of an affinely connected space .

Definition 3.1. A Finsler space F_n , whose coefficient connection parameter, $G_{jk}^{\,i}$ is independent of y^i is called an affinely connected space (Berwald space). Thus, an affinely connected space is characterized by any one of the following equivalent equations

(3.1) a) $G_{jkh}^{\,i} = 0$ and b) $C_{ijk|h} = 0$.

The coefficients connection parameters Γ_{kh}^{*i} of Cartan and $G_{kh}^{\,i}$ of Berwald coincide in affinely connected space and they are independent of directional argument [16], i.e.

(3.2) a) $\dot{\partial}_j G_{kh}^{\,i} = 0$ and b) $\dot{\partial}_j \Gamma_{kh}^{*i} = 0$.

Definition 3.2. The generalized R^h-fourecurrent space which possess the properties of an affinely connected space [satisfies any one of the equations (3.1a), (3.1b), (3.2a) and (3.2b)] will be called it a generalized R^h-fourecurrent affinely connected space and denoted briefly by R^h-G-FRF$_n$ - affinely connected space.

Remark 3.1. It will be sufficient to call Cartan's third curvature tensor R_{jkh}^i which possess the property of R^h-G-FRF$_n$ - affinely connected space as generalized h-forrecurrent tensor (briefly by R^h-G-FRF$_n$) .

Let us consider R^h-G-FRF$_n$ - affinely connected space.

In view of the theorem 2.1 and definition 3.2. , we may conclude

Theorem 3.1. In generalized R^h-recurrent a ffinely connected space, the generalized R^h- birecurrent affinely connected is R^h-G-FRF$_n$- affinely connected space.

Using (3.2b) in (2.12), we get

(3.3) $H_{jkh|l|m|n|s}^i - \{ (\dot{\partial}_r H_{kh}^i) P_{jl}^r \}_{|m|n|s} - \{ (\dot{\partial}_r H_{kh|l}^i) P_{jm}^r \}_{|n|s} - [\dot{\partial}_r \left(H_{kh|\ell|m}^i\right) P_{jn}^r]_{|s}$

$- [\left\{\dot{\partial}_r \left(H_{kh|\ell|m|n}^i\right)\right\}_{|s} - \dot{\partial}_q (H_{kh|\ell|m}^i) P_{rn}^q] P_{js}^r = \left(\dot{\partial}_j c_{\ell mns}\right) H_{kh}^i + c_{\ell mns} H_{jkh}^i$

$+ \left(\dot{\partial}_j d_{\ell mns}\right) \left(\delta_k^i y_h - \delta_h^i y_k\right) + d_{\ell mns} (\delta_k^i g_{jh} - \delta_h^i g_{jk})$.

This shows that

(3.4) $H_{jkh|\ell|m|n}^i = c_{\ell mn} H_{jkh}^i + d_{\ell mn} (\delta_k^i g_{jh} - \delta_h^i g_{jk})$,

if and only if

(3.5) $\{ (\dot{\partial}_r H_{kh}^i) P_{jl}^r \}_{|m|n|s} + \{ (\dot{\partial}_r H_{kh|l}^i) P_{jm}^r \}_{|n|s} + [\dot{\partial}_r \left(H_{kh|\ell|m}^i\right) P_{jn}^r]_{|s} + [\left\{\dot{\partial}_r \left(H_{kh|\ell|m|n}^i\right)\right\}_{|s}$

$+ \dot{\partial}_q (H_{kh|\ell|m}^i) P_{rn}^q] P_{js}^r + \left(\dot{\partial}_j c_{\ell mns}\right) H_{kh}^i + \left(\dot{\partial}_j d_{\ell mns}\right) \left(\delta_k^i y_h - \delta_h^i y_k\right) = 0$.

Further, using (3.2b) in (2.12), we get

(3.6) $H_{jpkh|\ell|m|n|s} - \{ g_{ip} (\dot{\partial}_r H_{kh}^i) P_{jl}^r \}_{|m|n|s} - \{ g_{ip} (\dot{\partial}_r H_{kh|l}^i) P_{jm}^r \}_{|n|s}$

$- [g_{ip}\, \dot{\partial}_r \left(H_{kh|\ell|m}^i\right) P_{jn}^r]_{|s} - [\left\{ g_{ip} \dot{\partial}_r \left(H_{kh|\ell|m|n}^i\right)\right\}_{|s} - g_{ip}\, \dot{\partial}_q (H_{kh|\ell|m}^i) P_{rn}^q] P_{js}^r$

$= \left(\dot{\partial}_j c_{\ell mns}\right) H_{kph} + c_{\ell mns} H_{jpkh} + g_{ip} \left(\dot{\partial}_j d_{\ell mns}\right) \left(\delta_k^i y_h - \delta_h^i y_k\right)$

$+ d_{\ell mns} (g_{kp}\, g_{jh} - g_{hp} g_{jk})$.

This shows that

(3.7) $H_{jpkh|\ell|m|n|s} = c_{\ell mns}\, H_{jpkh} + d_{\ell mns}(\, g_{kp}\, g_{jh} - g_{hp}\, g_{jk}\,)$

if and only if

(3.8) $\{\, g_{ip}\, (\dot{\partial}_r H^i_{kh})\, P^r_{jl}\,\}_{|m|n|s} + \{\, g_{ip}\, (\dot{\partial}_r H^i_{kh|l})\, P^r_{jm}\}_{|n|s} + [\, g_{ip}\, \dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_{jn}]_{|s}$
$+[\left\{ g_{ip}\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} + g_{ip}\, \dot{\partial}_q\, (H^i_{kh|\ell|m})\, P^q_{rn}\,]\, P^r_{js} = (\dot{\partial}_j c_{\ell mns})\, H_{kph}$
$+\, g_{ip}\, (\dot{\partial}_j d_{\ell mns})(\, \delta^i_k y_h - \delta^i_h y_k) = 0$.

Thus, we may conclude

Theorem 3.2. In R^h-G-FRF_n- affinely connected space H^i_{kjh} and its associative H_{jpkh} curvature tensor are generalized trirecurrent tensor if and only if (3.4) and (3.7), respectively hold good.

Transvecting (3.3) by y^j , using (1.5a), (1.7a), (1,9b), (1.20) and (1.19a), we get

(3.9) $H^i_{kh|l|m|n|s} - \{\, (\dot{\partial}_r H^i_{kh}) P^r_l\,\}_{|m|n|s} - \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} - [\, \dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n\,]_{|s}$
$-\, [\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s = (\dot{\partial}_j c_{\ell mns})\, H^i_{kh}\, y^j + c_{\ell mns}\, H^i_{kh}$
$+\, (\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)\, y^j + d_{\ell mns}(\delta^i_k y_h - \delta^i_h y_k)$.

This shows that

(3.10) $H^i_{kh|\ell|m|n|s} = c_{\ell mns} H^i_{kh} + d_{\ell mns}(\, \delta^i_k y_h - \delta^i_h y_k)$

if and only if

(3.11) $\{(\dot{\partial}_r H^i_{kh}) P^r_l\}_{|m|n|s} + \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} + [\dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n]_{|s} +[\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s}$
$+\dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s + (\dot{\partial}_j c_{\ell mns})\, H^i_{kh}\, y^j + (\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)\, y^j = 0$.

Transvecting (3.9) by g_{ir} , using (1.3a), (1.9a) and (1.19b), we get

(3.12) $H_{krh|l|m|n|s} - g_{ir}\{(\dot{\partial}_r H^i_{kh}) P^r_l\}_{|m|n|s} - \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} - [\, g_{ir}\dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n]_{|s}$
$-\, [\, g_{ir}\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s = (\dot{\partial}_j c_{\ell mns})\, H_{krh}\, y^j + c_{\ell mns}\, H_{krh}$
$+\, g_{ir}(\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)\, y^j + d_{\ell mns}(\, g_{kr} y_h - g_{hr} y_k)$.

This shows that

(3.13) $H_{krh|\ell|m|n|s} = c_{\ell mns} H_{krh} + d_{\ell mns}(\, g_{kr} y_h - g_h y_k\,)$

if and only if

(3.14) $g_{ir}\{(\dot{\partial}_r H^i_{kh}) P^r_l\}_{|m|n|s} + \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} + [\, g_{ir}\dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n]_{|s}$
$+\, [\, g_{ir}\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s + (\dot{\partial}_j c_{\ell mns})\, H_{krh}\, y^j$
$+\, g_{ir}(\dot{\partial}_j d_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)\, y^j = 0$.

Transvecting (3.9) by y^k , using (1.5a), (1.6a), (1.7a) and (1.19a), we get

(3.15) $H^i_{h|l|m|n|s} - y^k\{\, (\dot{\partial}_r H^i_{kh}) P^r_l\,\}_{|m|n|s} - \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} - y^k[\, \dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n\,]_{|s}$
$-\, y^k[\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s = (\dot{\partial}_j c_{\ell mns})\, H^i_h\, y^j + c_{\ell mns}\, H^i_h$
$+\, (\dot{\partial}_j d_{\ell mns})(\, y_h\, y^k - \delta^i_h F^2\,)\, y^j + d_{\ell mns}(\, y_h\, y^k - \delta^i_h F^2\,)$.

This shows that

(3.16) $H^i_{h|l|m|n|s} = c_{\ell mns} H^i_h + d_{\ell mns}(\, y_h\, y^k - \delta^i_h F^2\,)$

if and only if

(3.17) $y^k\{\, (\dot{\partial}_r H^i_{kh}) P^r_l\,\}_{|m|n|s} + \{(\dot{\partial}_r H^i_{kh|l}) P^r_m\}_{|n|s} + y^k[\, \dot{\partial}_r \left(H^i_{kh|\ell|m}\right) P^r_n\,]_{|s}$
$+\, y^k[\left\{\dot{\partial}_r \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H^i_{kh|\ell|m}) P^q_{rn}]\, P^r_s + (\dot{\partial}_j c_{\ell mns})\, H^i_h\, y^j$
$+\, (\dot{\partial}_j d_{\ell mns})(\, y_h\, y^k - \delta^i_h F^2\,)\, y^j = 0$.

Thus, we may conclude

Theorem 3.3. In R^h-G-FRF_n- affinely connected space, the h-covariant derivative of fourth order for the h(v)-torsion tensor H^i_{kh} , its associative tensor H_{krh} and the deviation tensor H^i_h given by (3.10) ,(3.13) and (3.16) if and only if (3.11), (3.14) and (3.17), respectively hold.

Contracting the indices i and h in (3.3), using (1.13c), (1.14c), (1.7b) and (1.4), we get

(3.18) $H_{jk|l|m|n|s} - \{(\dot{\partial}_r H_k\,) P^r_{jl}\,\}_{|m|n|s} - \{(\dot{\partial}_r H_{k|l}) P^r_{jm}\}_{|n|s} - [\, \dot{\partial}_r \left(H_{k|\ell|m}\right) P^r_{jn}\,]_{|s}$
$-\, [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H_{k|\ell|m}) P^q_{rn}]\, P^r_{js} = (\dot{\partial}_j c_{\ell mns}) H_k + c_{\ell mns}\, H_{jk}$
$+\, (\,1-n\,)\, (\dot{\partial}_j d_{\ell mns})\, y_k + (\,1-n\,) d_{\ell mns}\, g_{jk}$.

This shows that

(3.19) $H_{jk|l|m|n|s} = c_{\ell mns}\, H_{jk} + (1-n)\, d_{\ell mns}\, g_{jk}$

if and only if

(3.20) $\{(\dot{\partial}_r H_k\,) P^r_{jl}\,\}_{|m|n|s} + \{(\dot{\partial}_r H_{k|l}) P^r_{jm}\}_{|n|s} + [\, \dot{\partial}_r \left(H_{k|\ell|m}\right) P^r_{jn}\,]_{|s} + [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s}$
$-\, \dot{\partial}_q (H_{k|\ell|m}) P^q_{rn}\,]\, P^r_{js} + (\dot{\partial}_j c_{\ell mns}) H_k = 0$.

Contracting the indices i and h in (3.9), using (1.13c) , (1.7b) and (1.4), we get

(3.21) $H_{k|l|m|n|s} - \{(\dot{\partial}_r H_k\,) P^r_l\,\}_{|m|n|s} - \{(\dot{\partial}_r H_{k|l}) P^r_m\}_{|n|s} - [\, \dot{\partial}_r \left(H_{k|\ell|m}\right) P^r_n]_{|s}$
$-\, [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s} - \dot{\partial}_q (H_{k|\ell|m})\, P^q_{rn}]\, P^r_s = (\dot{\partial}_j c_{\ell mns})\, H_k\, y^j + c_{\ell mns}\, H_k$
$+(1-n)(\dot{\partial}_j d_{\ell mns})\, y_k\, y^j + (1-n)\, d_{\ell mns}\, y_k$.

This shows that

(3.22) $H_{k|l|m|n|s} = c_{\ell mns}\, H_k + (1-n)\, d_{\ell mns}\, y_k$

if and only if

(3.23) $\{\, (\dot{\partial}_r H_k\,) P^r_l\,\}_{|m|n|s} + \{(\dot{\partial}_r H_{k|l}) P^r_m\}_{|n|s} + [\, \dot{\partial}_r \left(H_{k|\ell|m}\right) P^r_n]_{|s} + [\left\{\dot{\partial}_r \left(H_{k|\ell|m|n}\right)\right\}_{|s}$

$-\dot{\partial}_q (H_{k|\ell|m})\, P^q_{rn}]\, P^r_s + (\dot{\partial}_j c_{\ell mns})\, H_k\, y^j + (1-n)(\dot{\partial}_j d_{\ell mns})\, y_k\, y^j = 0$.

Transvecting (3.21) by y^k *, using (1.5a) and (1.15), we get*

(3.24) $H_{|l|m|n|s} - \{(\dot{\partial}_r H)P^r_l\}_{|m|n|s} - \{(\dot{\partial}_r H_{|l})P^r_m\}_{|n|s} - [\,\dot{\partial}_r (H_{|\ell|m})\, P^r_n]_{|s}$

$- [\{\dot{\partial}_r (H_{|\ell|m|n})\}_{|s} - \dot{\partial}_q (H_{|\ell|m})\, P^q_{rn}]\, P^r_s = (\dot{\partial}_j c_{\ell mns})\, H\, y^j + c_{\ell mns}\, H$

$+(1-n)(\dot{\partial}_j d_{\ell mns})\, F^2\, y^j + (1-n)\, d_{\ell mns}\, F^2$.

This shows that

(3.25) $H_{|l|m|n|s} = c_{\ell mns}\, H + d_{\ell mns}\, F^2$

if and only if

(3.26) $\{(\dot{\partial}_r H)P^r_l\}_{|m|n|s} + \{(\dot{\partial}_r H_{|l})P^r_m\}_{|n|s} + [\,\dot{\partial}_r (H_{|\ell|m})\, P^r_n]_{|s} + [\{\dot{\partial}_r (H_{|\ell|m|n})\}_{|s}$

$-\dot{\partial}_q (H_{|\ell|m})\, P^q_{rn}]\, P^r_s + (\dot{\partial}_j c_{\ell mns})\, H\, y^j + (\dot{\partial}_j d_{\ell mns})\, F^2\, y^j = 0$.

The equations (3.19), (3.22) and (3.25) show that the H-Ricci tensor H_{jk} , the curvature vector H_k and the scalar curvature H, can't vanish, because the vanishing of any one of them would imply $d_{\ell mn} = 0$, if and only if (3.20), (3.23) and (3.26), respectively, hold, a contradiction.

Thus, we may conclude

Theorem 3.4. In R^h-G-FRF_n- affinely connected space, the H-Ricci tensor H_{jk} , the curvature vector H_k and the scalar curvature H, are non-vanishing if and only if (3.20), (3.23) and (3.26), respectively hold.

IV. Conclusions

(4.1) The R^h-generalized fourecurrent space is called R^h-generalized fourecurrent affinely connected if it satisfies any one of the conditions (3.1a), (3.1b), (3.2a) and (3.2b).

(4.2) In R^h -affinely connected space, if the directional derivative of covariant vector field and covariant tensor of third order are vanish, then Berwald curvature tensor H^i_{jkh} is generalized fourecurrent.

(4.3) In R^h -affinely connected space, if the directional derivative of covariant vector field and covariant tensor of fourth order are vanish, then h(v)-torsion tensor H^i_{kh} , the deviation tensor H^i_k, the curvature vector H_k , the curvature scalar H and the tensor H_{kph} are all generalized fourecurrent.

(4.4) In R^h-G-FRF_n- affinely connected space, Ricci tensor H_{jk} in sense of Berwald coincide with Ricci tensor R_{jk} of Cartan's fourth curvature.

(4.5) In R^h-G-FRF_n- affinely connected space the associate curvature tensor H_{jpkh} of Berwald curvature tensor coincide with the associate curvature tensor R_{jpkh} Cartan's fourth curvature tensor.

V. Recommendations

Authors recommend the need for the continuing research and development in Finsler space due to its vital applying importance in other fields.

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