New Product Binary Operations on Graphs

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Abstract

In this paper, we introduce new binary operations on graphs. In fact, we obtained some other product operations, called them classic product operations from union of two or more new product operations. We examined the relationship between new binary product operations and classic product operations.

Keywords: Graph,Complete graph, Complement graph, Graph operations.

1. Introduction

A graph G consists of a non-empty set of elements called vertices and a list of unordered pairs of these elements called edges. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Throughout this paper, we consider finite graphs that have no loops or multiple edges. The degree of the vertex (v) is the number of edges joined with this vertex is denoted by $\delta(v)$. The notion $|V|$ and $|E|$ are used to indicate the number of vertices and edges respectively [2].

Let G be simple graph with p vertices. The complement graph G^c of G is defined to be the simple graph with the same vertex set as G and where two vertices u and v are adjacent precisely when they are not adjacent in G. Roughly speaking then, the complement of G can be obtained from the complete graph K_p by "rubbing out" all the edges of G [1].

A product binary operation, creates a new graph from two initial graphs, some binary operations called them Elementary Binary Operations, They create a new graph from two initial graphs by change of vertices or edges or both such as union or join, some other binary operations called them product binary operations,They also create a new graph from two initial graphs, where the resulting graph has the same set of vertices but its set of edges depends of the considered operation, such as tensor product, cartesian product, strong product, composition, symmetric difference and disjunction [5].

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1.1 Definition: If G_1 and G_2 be two simple connected graphs, then

(a) The vertices setsare defined as follows [3]:

$V(G_1 \square G_2) = V(G_1) \times V(G_2)$, where $\square \in \{\otimes, \times, *, \circ, \oplus, \vee\}$

(b) The edges sets are defined as follows [3]:

(1) $E(G_1 \otimes G_2) = \{(a,b)(c,d) : [ac \in E(G_1), bd \in E(G_2)]\}$

(2) $E(G_1 \times G_2) = \{(a,b)(c,d) : [ac \in E(G_1), b = d]$ or $[bd \in E(G_2), a = c]\}$

(3) $E(G_1 * G_2) = \{(a,b)(c,d) : [ac \in E(G_1), b = d] or [bd \in E(G_2), a = c] or [ac \in E(G_1), bd \in E(G_2)]\}$

(4) $E(G_1 \circ G_2) = \{(a,b)(c,d) : [ac \in E(G_1)] or [bd \in E(G_2), a = c]\}$

(5) $E(G_1 \oplus G_2) = \{(a,b)(c,d) : [ac \in E(G_1)] or [bd \in E(G_2)]\}$ but not both

(6) $E(G_1 \vee G_2) = \{(a,b)(c,d) : [ac \in E(G_1)] or [bd \in E(G_2)]\}$

For convenience, we will call the cartesian product $G_1 \times G_2$, strong product $G_1 * G_2$, composition $G_1 \circ G_2$, symmetric difference $G_1 \oplus G_2$ and disjunction $G_1 \vee G_2$ classic product operations.

1.2 Lemma: Consider two graphs G_1 and G_2 where

$|V(G_1)| = p_1, |V(G_2)| = p_2, |E(G_1)| = q_1$ and $|E(G_2)| = q_2$

(a) The number of vertices sets are equals [4]:

$|V(G_1 \square G_2)| = p_1 p_2$, where $\square \in \{\otimes, \times, *, \circ, \oplus, \vee\}$

(b) The number of edges sets are equals [4]:

(1) $|E(G_1 \otimes G_2)| = 2q_1 q_2$

(2) $|E(G_1 \times G_2)| = q_1 p_2 + q_2 p_1$

(3) $|E(G_1 * G_2)| = q_1 p_2 + q_2 p_1 + 2q_1 q_2$

(4) $|E(G_1 \circ G_2)| = q_1 p_2^2 + q_2 p_1$

(5) $|E(G_1 \oplus G_2)| = q_1 p_2^2 + q_2 p_1^2 - 4q_1 q_2$

(6) $|E(G_1 \vee G_2)| = q_1 p_2^2 + q_2 p_1^2 - 2q_1 q_2$

In this paper, we symbolize with some new product operations on graphs, denoted $\otimes_i$, where $i \in \{1, 2, ..., 7\}$, and defined as follows:

1.3 Definition: If G_1 and G_2 be two simple connected graphs, then

(a) The vertices sets are defined as follows:

$$V(G_1 \otimes_i G_2) = V(G_2 \otimes_i G_1) = V(G_1) \times V(G_2)$$

(b) The edges sets are defined as follows:

(1) $E(G_1 \otimes_1 G_2) = \{(a,b)(c,d) : [ac \in E(G_1), b = d]\}$

(2) $E(G_1 \otimes_2 G_2) = \{(a,b)(c,d) : [ac \in E(G_1), bd \in E(G_2^c)]\}$

(3) $E(G_1 \otimes_3 G_2) = (a,b)(c,d) : [a = c, bd \in E(G_2)]$

(4) $E(G_1 \otimes_4 G_2) = \{(a,b)(c,d) : [a = c, bd \in E(G_2^c)]\}$

(5) $E(G_1 \otimes_5 G_2) = \{(a,b)(c,d) : [ac \in E(G_1^c), bd \in E(G_2)]\}$

(6) $E(G_1 \otimes_6 G_2) = \{(a,b)(c,d) : [ac \in E(G_1^c), b = d]\}$

(7) $E(G_1 \otimes_7 G_2) = \{(a,b)(c,d) : [ac \in E(G_1^c), bd \in E(G_2^c)]\}$

Along this line we fund that the tensor product operation play a prominent role in the sequel. For convenience we consider this operation is a new product operation and we will denote this operator by $\otimes_0$ rather $\otimes$. Any other unexplained terminology is standard as in [5-10].

2. Properties for a new product operations

In this section we will compute the properties for a new product operations

2.1 Lemma: Consider two graphs G_1 and G_2 where; $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ The number of vertex sets are equals:

$$|V(G_1 \otimes_i G_2)| = p_1 p_2 : i = 1, 2, ..., 7$$

Proof: By definition 1.3 we have

$$V(G_1 \otimes_i G_2) = V(G_1) \times V(G_2)$$

then it's easy to see that

$$|V(G_1 \otimes_i G_2)| = |V(G_1)| \times |V(G_2)| = p_1 p_2. \quad \square$$

2.2 Lemma: Consider two graphs G_1 and G_2 where; $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$ The number of edge sets are equals:

(1) $|E(G_1 \otimes_1 G_2)| = q_1 p_2$

(2) $|E(G_1 \otimes_2 G_2)| = 2q_1 q_2^c = p_2^2 q_1 - p_2 q_1 - 2q_1 q_2$

(3) $|E(G_1 \otimes_3 G_2)| = q_2 p_1$

(4) $|E(G_1 \otimes_4 G_2)| = q_2^c p_1 = \frac{1}{2}(p_1 p_2^2 - p_1 p_2 - 2p_1 q_2)$

(5) $|E(G_1 \otimes_5 G_2)| = 2q_1^c q_2 = p_1^2 q_2 - p_1 q_2 - 2q_1 q_2$

(6) $|E(G_1 \otimes_6 G_2)| = q_1^c p_2 = \frac{1}{2}(p_2 p_1^2 - p_1 p_2 - 2p_2 q_1)$

(7) $|E(G_1 \otimes_7 G_2)| = 2q_1^c q_2^c = \frac{1}{2}[p_1(p_1 - 1) - 2q_1][p_2(p_2 - 1) - 2q_2]$

Proof: We have $G_1 \cup G_1^c = K_{p_1}$ where K_{p_1} be complete graph with p_1 vertices, then

$$|E(G_1 \cup G_1^c)| = |E(K_{p_1})| = \binom{p_1}{2}$$

It follows that

$$|E(G_1)| + |E(G_1^c)| = \binom{p_1}{2}$$

that is

$$q_1 + q_1^c = \binom{p_1}{2}$$

Therefore

$$q_i^c = \binom{p_i}{2} - q_i \ : i = 1,2$$

1. We have $|E(G_1 \otimes_1 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1), (v_1 = v_2)\}$

For a fixed vertex v in $V(G_2)$, there exists an edge between (u_1,v) and (u_2,v), where u_1,u_2 are adjacent in G_1. Theref ore, $|E(G_1 \otimes_1 G_2)| = |E(G_1)| \cdot |V(G_2)| = q_1 p_2$.

2. We have $|E(G_1 \otimes_2 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1), (v_1v_2 \in E_2^c)\}$

Note that $|E(G_1 \otimes_2 G_2)| = |E(G_1 \otimes_0 G_2^c)|$

So $|E(G_1 \otimes_2 G_2)| = 2|E(G_1)| \cdot |E(G_2^c)|$

It follows that $|E(G_1 \otimes_2 G_2)| = 2[q_1][\frac{1}{2}p_2(p_2-1) - q_2] = [q_1][p_2(p_2-1) - 2q_2]$,

that is $|E(G_1 \otimes_2 G_2)| = p_2^2 q_1 - p_2 q_1 - 2q_1 q_2$.

3. We have $|E(G_1 \otimes_3 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1 = u_2), (v_1v_2 \in E_2)\}$

By writing $|E(G_1 \otimes_3 G_2)| = |E(G_2 \otimes_1 G_1)|$,

we find that $|E(G_1 \otimes_3 G_2)| = q_2 p_1$.

4. We have $|E(G_1 \otimes_4 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1 = u_2), (v_1v_2 \in E_2^c)\}$

We can write $|E(G_1 \otimes_4 G_2)| = |E(G_2^c \otimes_1 G_1)| = |E(G_2^c)| \cdot |V(G_1)|$

It follows that $|E(G_1 \otimes_4 G_2)| = \frac{1}{2}(p_1 p_2^2 - p_1 p_2 - 2p_1 q_2)$.

5. We have $|E(G_1 \otimes_5 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1^c), (v_1v_2 \in E_2)\}$.

As $|E(G_1 \otimes_5 G_2)| = 2|E(G_1^c)| \cdot |E(G_2)|$

then $|E(G_1 \otimes_5 G_2)| = p_1^2 q_2 - p_1 q_2 - 2q_1 q_2$

6. We have $|E(G_1 \otimes_6 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1^c), (v_1 = v_2)\}$

We can write $|E(G_1 \otimes_6 G_2)| = |E(G_1^c \otimes_1 G_2)| = |E(G_1^c)| \cdot |V(G_2)|$

It follows that $|E(G_1 \otimes_6 G_2)| = \frac{1}{2}(p_2 p_1^2 - p_1 p_2 - 2p_2 q_1)$

7. We have $|E(G_1 \otimes_7 G_2)| = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1^c),(v_1v_2 \in E_2^c)\}$

We can write $|E(G_1 \otimes_7 G_2)| = |E(G_1^c \otimes_0 G_2^c)| = 2|E(G_1^c)| \cdot |E(G_2^c)|$

It follows that $|E(G_1 \otimes_7 G_2)| = \frac{1}{2}[p_1(p_1-1)-2q_1][p_2(p_2-1)-2q_2]$. □

2.3 Lemma: If G_1 and G_2 be two graphs where; $|V(G_1)| = p_1$, $|V(G_2)| = p_2, |E(G_1)| = q_1$ and $|E(G_2)| = q_2$, then

(1) $\delta_{G_1 \otimes_1 G_2}(u,v) = \delta_{G_1} u$

(2) $\delta_{G_1 \otimes_2 G_2}(u,v) = \delta_{G_1} u \, \delta_{G_2^c} v$

(3) $\delta_{G_1 \otimes_3 G_2}(u,v) = \delta_{G_2} v$

(4) $\delta_{G_1 \otimes_4 G_2}(u,v) = \delta_{G_2^c} v$

(5) $\delta_{G_1 \otimes_5 G_2}(u,v) = \delta_{G_1^c} u \, \delta_{G_2} v$

(6) $\delta_{G_1 \otimes_6 G_2}(u,v) = \delta_{G_1^c} u$

(7) $\delta_{G_1 \otimes_7 G_2}(u,v) = \delta_{G_1^c} u \, \delta_{G_2^c} v$

Proof: (1) We have $$\sum_{(u,v) \in V(G_1 \otimes_1 G_2)} \delta_{G_1 \otimes_1 G_2}(u,v) = 2|E(G_1 \otimes_1 G_2)| = 2q_1p_2$$

Therefore $$\sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1 \otimes_1 G_2}(u,v) = \sum_{u \in V(G_1)} \delta_{G_1} u \sum_{v \in V(G_2)} 1$$

We can write $$\sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1 \otimes_1 G_2}(u,v) = \sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1} u$$

It follows that $$\delta_{G_1 \otimes_1 G_2}(u,v) = \delta_{G_1} u$$

(2) We have $$\sum_{(u,v) \in V(G_1 \otimes_2 G_2)} \delta_{G_1 \otimes_2 G_2}(u,v) = 2|E(G_1 \otimes_2 G_2)| = 4q_1q_2^c$$

Therefore $$\sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1 \otimes_2 G_2}(u,v) = \sum_{u \in V(G_1)} \delta_{G_1} u \sum_{v \in V(G_2)} \delta_{G_2^c} v$$

We can write $$\sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1 \otimes_2 G_2}(u,v) = \sum_{(u,v) \in V(G_1 \times G_2)} \delta_{G_1} u \, \delta_{G_2^c} v$$

It follows that $$\delta_{G_1 \otimes_2 G_2}(u,v) = \delta_{G_1} u \, \delta_{G_2^c} v$$

(3) We have $$\sum_{(u,v)\in V(G_1\otimes_3 G_2)} \delta_{G_1\otimes_3 G_2}(u,v) = 2|E(G_1\otimes_3 G_2)| = 2q_2p_1$$

Therefore $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_3 G_2}(u,v) = \sum_{u\in V(G_1)} 1 \sum_{v\in V(G_2)} \delta_{G_2}v$$

We can write $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_3 G_2}(u,v) = \sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_2}v$$

It follows that $$\delta_{G_1\otimes_3 G_2}(u,v) = \delta_{G_2}v$$

(4) We have $$\sum_{(u,v)\in V(G_1\otimes_4 G_2)} \delta_{G_1\otimes_4 G_2}(u,v) = 2|E(G_1\otimes_4 G_2)| = 2q_2^c p_1$$

Therefore $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_4 G_2}(u,v) = \sum_{u\in V(G_1)} 1 \sum_{v\in V(G_2)} \delta_{G_2^c}v$$

We can write $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_4 G_2}(u,v) = \sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_2^c}v$$

It follows that $$\delta_{G_1\otimes_4 G_2}(u,v) = \delta_{G_2^c}v$$

(5) We have $$\sum_{(u,v)\in V(G_1\otimes_5 G_2)} \delta_{G_1\otimes_5 G_2}(u,v) = 2|E(G_1\otimes_5 G_2)| = 2(2q_1^c q_2)$$

Therefore $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_5 G_2}(u,v) = \sum_{u\in V(G_1)} \delta_{G_1^c}u \sum_{v\in V(G_2)} \delta_{G_2}v$$

We can write $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_5 G_2}(u,v) = \sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1^c}u\,\delta_{G_2}v$$

It follows that $$\delta_{G_1\otimes_2 G_5}(u,v) = \delta_{G_1^c}u\,\delta_{G_2}v$$

(6) We have $$\sum_{(u,v)\in V(G_1\otimes_6 G_2)} \delta_{G_1\otimes_6 G_2}(u,v) = 2|E(G_1\otimes_6 G_2)| = 2q_1^c p_2$$

Therefore $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_6 G_2}(u,v) = \sum_{u\in V(G_1)} \delta_{G_1^c}u \sum_{v\in V(G_2)} 1$$

We can write $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_6 G_2}(u,v) = \sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1^c}u$$

It follows that $$\delta_{G_1\otimes_6 G_2}(u,v) = \delta_{G_1^c}u$$

(7) We have $$\sum_{(u,v)\in V(G_1\otimes_7 G_2)} \delta_{G_1\otimes_7 G_2}(u,v) = 2|E(G_1\otimes_7 G_2)| = 2(2q_1^c q_2^c)$$

Therefore $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_7 G_2}(u,v) = \sum_{u\in V(G_1)} \delta_{G_1^c} u \sum_{v\in V(G_2)} \delta_{G_2^c} v$$

We can write $$\sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1\otimes_7 G_2}(u,v) = \sum_{(u,v)\in V(G_1\times G_2)} \delta_{G_1^c} u\, \delta_{G_2^c} v$$

It follows that $$\delta_{G_1\otimes_2 G_7}(u,v) = \delta_{G_1^c} u\, \delta_{G_2^c} v. \quad \square$$

2.4 Corollary: If G_1 and G_2 be two graphs, then we can see easy

1. All new product operations are noncommutative, except zero product operation $G_1 \otimes_0 G_2$ and seventh product operation $G_1 \otimes_7 G_2$.
2. All new product operations are associative, except second product operation $G_1 \otimes_2 G_2$ and fifth product operation $G_1 \otimes_5 G_2$.

2.5 Example: If G_1 and G_2 be two graphs follows in figure 1,

Then the graphs of a new product operations follows in figure 2.

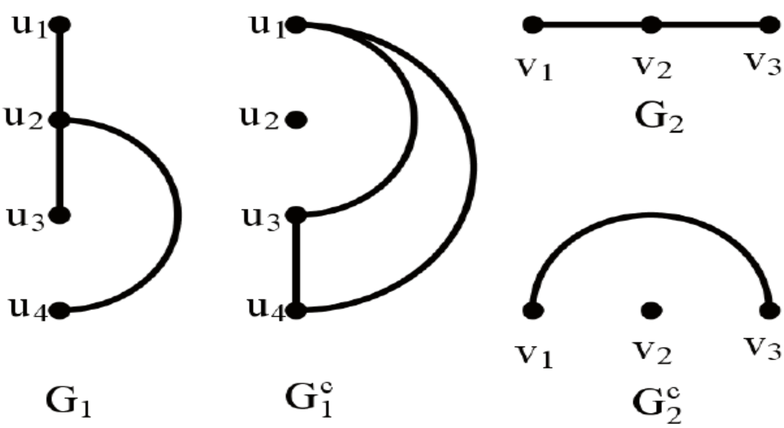

Figure (1): Two Graphs (G1-G2)

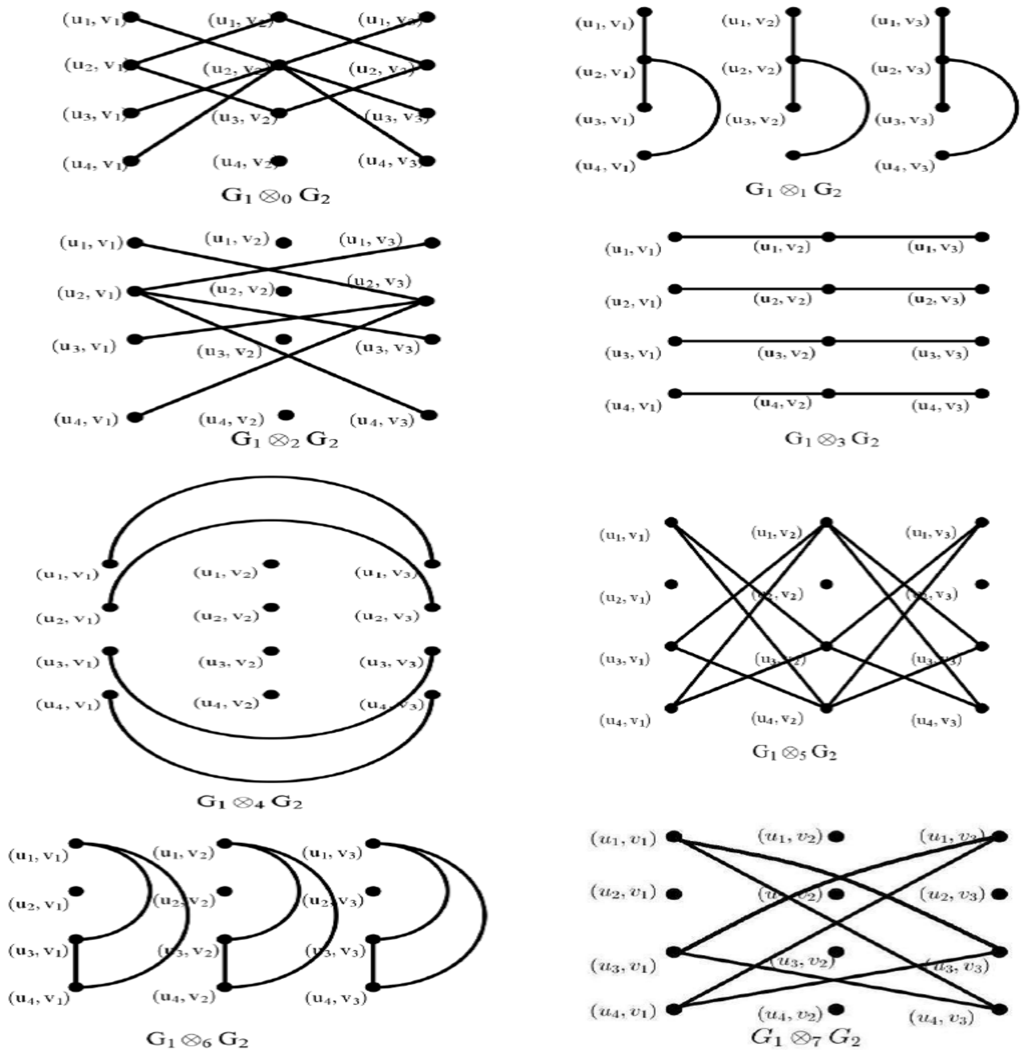

Figure (2): New Product Binary Operations

3. Comparison between new and classic operations

In this section, we investigate the comparison between our new product operations and classic product operations.

3.1 Theorem: If G_1 and G_2 are two simple and connected graphs, then

$$\bigcup_{i=1, i\neq 2}^{3} (G_1 \otimes_i G_2) = G_1 \times G_2$$

Proof: We have $V(G_1 \otimes_1 G_2) \cup (G_1 \otimes_3 G_2) = V(G_1 \times G_2) = V_1 \times V_2$

As $E(G_1 \otimes_1 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1 = v_2\}$,

$$E(G_1 \otimes_3 G_2) = \{(u_1,v_1)(u_2,v_2) : v_1v_2 \in E_2, u_1 = u_2\}.$$

Then $\bigcup_{i=1, i\neq 2}^{3} E(G_1 \otimes_i G_2) = \{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E_1, v_1 = v_2) or (v_1v_2 \in E_2, u_1 = u_2)\}$

It follows that $\bigcup_{i=1, i\neq 2}^{3} E(G_1 \otimes_i G_2) = E(G_1 \times G_2)$

Hence

$$\bigcup_{i=1, i\neq 2}^{3} (G_1 \otimes_i G_2) = G_1 \times G_2. \quad \square$$

3.2 Theorem: If G_1 and G_2 are two simple and connected graphs, then

$$\bigcup_{i=0, i\neq 2}^{3} (G_1 \otimes_i G_2) = G_1 * G_2$$

Proof: We have

$$\bigcup_{i=0, i\neq 2}^{3} V(G_1 \otimes_i G_2) = V_1 \times V_2$$

As

$$E(G_1 \otimes_0 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\},$$
$$E(G_1 \otimes_1 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1 = v_2\},$$
$$E(G_1 \otimes_3 G_2) = \{(u_1,v_1)(u_2,v_2) : v_1v_2 \in E_2, u_1 = u_2\}.$$

Then

$$\bigcup_{i=0, i\neq 2}^{3} E(G_1 \otimes_i G_2) = (u_1,v_1)(u_2,v_2) : \begin{cases} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ u_1u_2 \in E_1, v_1 = v_2 \\ v_1v_2 \in E_2, u_1 = u_2. \end{cases}$$

It follows that

$$\bigcup_{i=0, i\neq 2}^{3} E(G_1 \otimes_i G_2) = E(G_1 * G_2)$$

Hence

$$\bigcup_{i=0, i\neq 2}^{3} (G_1 \otimes_i G_2) = G_1 * G_2. \quad \square$$

3.3 Theorem: If G_1 and G_2 are two simple and connected graphs, then

$$\bigcup_{i=1}^{3} (G_1 \otimes_i G_2) = G_1 \circ G_2$$

Proof: We have

$$\bigcup_{i=0}^{3} V(G_1 \otimes_i G_2) = V_1 \times V_2$$

As

$$E(G_1 \otimes_0 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\},$$
$$E(G_1 \otimes_1 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1 = v_2\},$$
$$E(G_1 \otimes_2 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2^c\},$$
$$E(G_1 \otimes_3 G_2) = \{(u_1,v_1)(u_2,v_2) : v_1v_2 \in E_2, u_1 = u_2\}.$$

Then
$$\bigcup_{i=0}^{3} E(G_1 \otimes_i G_2) = (u_1,v_1)(u_2,v_2) : \begin{cases} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ u_1u_2 \in E_1, v_1 = v_2 \\ u_1u_2 \in E_1, v_1v_2 \in E_2^c \\ v_1v_2 \in E_2, u_1 = u_2. \end{cases}$$

It follows that
$$\bigcup_{i=0}^{3} E(G_1 \otimes_i G_2) = E(G_1 \circ G_2)$$

Hence
$$\bigcup_{i=0}^{3} (G_1 \otimes_i G_2) = G_1 \circ G_2. \quad \square$$

3.4 Theorem: If G_1 and G_2 are two simple and connected graphs, then
$$\bigcup_{i=1, i\neq 4}^{5} (G_1 \otimes_i G_2) = G_1 \oplus G_2$$

Proof: We have
$$\bigcup_{i=1, i\neq 4}^{5} V(G_1 \otimes_i G_2) = V_1 \times V_2$$

As
$$E(G_1 \otimes_1 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1 = v_2\},$$
$$E(G_1 \otimes_2 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2^c\},$$
$$E(G_1 \otimes_3 G_2) = \{(u_1,v_1)(u_2,v_2) : v_1v_2 \in E_2, u_1 = u_2\},$$
$$E(G_1 \otimes_5 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1^c, v_1v_2 \in E_2\}.$$

Then
$$\bigcup_{i=1, i\neq 4}^{5} E(G_1 \otimes_i G_2) = (u_1,v_1)(u_2,v_2) : \begin{cases} u_1u_2 \in E_1, v_1 = v_2 \\ u_1u_2 \in E_1, v_1v_2 \in E_2^c \\ v_1v_2 \in E_2, u_1 = u_2 \\ u_1u_2 \in E_1^c, v_1v_2 \in E_2. \end{cases}$$

It follows that
$$\bigcup_{i=1, i\neq 4}^{5} E(G_1 \otimes_i G_2) = E(G_1 \oplus G_2)$$

Hence
$$\bigcup_{i=1, i\neq 4}^{5} (G_1 \otimes_i G_2) = G_1 \oplus G_2. \quad \square$$

3.5 Theorem: If G_1 and G_2 are two simple and connected graphs, then
$$\bigcup_{i=0, i\neq 4}^{5} (G_1 \otimes_i G_2) = G_1 \vee G_2$$

Proof: We have
$$\bigcup_{i=0, i\neq 4}^{5} V(G_1 \otimes_i G_2) = V_1 \times V_2$$

As
$$E(G_1 \otimes_0 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\},$$
$$E(G_1 \otimes_1 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1 = v_2\},$$
$$E(G_1 \otimes_2 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2^c\},$$
$$E(G_1 \otimes_3 G_2) = \{(u_1,v_1)(u_2,v_2) : v_1v_2 \in E_2, u_1 = u_2\},$$

$$E(G_1 \otimes_5 G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1^c, v_1v_2 \in E_2\}.$$

Then
$$\bigcup_{i=0, i\neq 4}^{5} E(G_1 \otimes_i G_2) = (u_1,v_1)(u_2,v_2) : \begin{cases} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ u_1u_2 \in E_1, v_1 = v_2 \\ u_1u_2 \in E_1, v_1v_2 \in E_2^c \\ v_1v_2 \in E_2, u_1 = u_2 \\ u_1u_2 \in E_1^c, v_1v_2 \in E_2. \end{cases}$$

It follows that
$$\bigcup_{i=0, i\neq 4}^{5} E(G_1 \otimes_i G_2) = E(G_1 \vee G_2)$$

Hence
$$\bigcup_{i=0, i\neq 4}^{5} (G_1 \otimes_i G_2) = G_1 \vee G_2. \quad \square$$

4. Some Remarks on new product operations

In this section, we provide some remarks on new product operations.

4.1 Remark: Any classic product operation can be deduced from our new product operations

Proof: By definition 1.1 and definition 1.3, we have

(1) $G_1 \times G_2 = \bigcup_{i=1, i\neq 2}^{3} (G_1 \otimes_i G_2)$

(2) $G_1 * G_2 = \bigcup_{i=0, i\neq 2}^{3} (G_1 \otimes_i G_2)$

(3) $G_1 \circ G_2 = \bigcup_{i=0}^{3} (G_1 \otimes_i G_2)$

(4) $G_1 \oplus G_2 = \bigcup_{i=1, i\neq 4}^{5} (G_1 \otimes_i G_2)$

(5) $G_1 \vee G_2 = \bigcup_{i=0, i\neq 4}^{5} (G_1 \otimes_i G_2)$

4.2.Remark: Any operation of new product operations can be generated from only zero and first product operations.

Proof: By Lemma 2.1 and Theorems (3.1-3.5), we get

(1) $G_1 \otimes_3 G_2 = G_2 \otimes_1 G_1$

(2) $G_1 \otimes_4 G_2 = G_2^c \otimes_1 G_1$

(3) $G_1 \otimes_5 G_2 = G_1^c \otimes_0 G_2$

(4) $G_1 \otimes_6 G_2 = G_1^c \otimes_1 G_2$

(5) $G_1 \otimes_7 G_2 = G_1^c \otimes_0 G_2^c$

DOI: 10.20428/JST.21.1.7

4.3 Corollary: Any classicproduct operation can be generated from only zero and first product operations.

Proof: By Remark 4.1 and Remark 4.2, we get

(1) $G_1 \times G_2 = (G_1 \otimes_1 G_2) \cup (G_2 \otimes_1 G_1)$

(2) $G_1 * G_2 = (G_1 \otimes_0 G_2) \cup (G_1 \otimes_1 G_2) \cup (G_2 \otimes_1 G_1)$

(3) $G_1 \circ G_2 = (G_1 \otimes_0 G_2) \cup (G_1 \otimes_1 G_2) \cup (G_1 \otimes_0 G_2^c) \cup (G_2 \otimes_1 G_1)$

(4) $G_1 \oplus G_2 = (G_1 \otimes_1 G_2) \cup (G_1 \otimes_0 G_2^c) \cup (G_2 \otimes_1 G_1) \cup (G_1^c \otimes_0 G_2)$

(5) $G_1 \vee G_2 = (G_1 \otimes_0 G_2) \cup (G_1 \otimes_1 G_2) \cup (G_1 \otimes_0 G_2^c) \cup (G_2 \otimes_1 G_1) \cup (G_1^c \otimes_0 G_2)$

4.4 Remark : For $i, j \in \{0,1,...,7\}$

$$E(G_1 \otimes_i G_2) \cap E(G_1 \otimes_j G_2) = \phi \quad \text{where } i \neq j$$

Proof: To prove that we use truth tables, we assume that

$(p = u_1u_2 \in E_1), (q = u_1 = u_2), (s = v_1v_2 \in E_2), (t = v_1 = v_2)$

We obtain $(2^4 = 16)$ possibilities in the truth table,

Table (1): Truth Table

	p	q	s	t
1	$u_1u_2 \in E_1$	$u_1 = u_2$	$v_1v_2 \in E_2$	$v_1 = v_2$
2	$u_1u_2 \in E_1$	$u_1 = u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
3	$u_1u_2 \in E_1$	$u_1 = u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
4	$u_1u_2 \in E_1$	$u_1 = u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$
5	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 = v_2$
6	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
7	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
8	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$
9	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \in E_2$	$v_1 = v_2$
10	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
11	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
12	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$

	p	q	s	t
13	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 = v_2$
14	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
15	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
16	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$

We omit eight cases, because of impossible probabilities for instant, the first case is cancelled, because there is no edge in simple graphs of the type $(u_1u_2 \in E_1 : u_1 = u_2)$. Similarly we omit the cases two, three, four, five, nine and thirteen. For the same result, we omit the eleven case there leads to a trivial graph. Consequently, we obtain eight different cases, that we illustrate in table (2).

Table (2): Real Truth

	p	q	s	t
1	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
2	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
3	$u_1u_2 \in E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$
4	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
5	$u_1u_2 \notin E_1$	$u_1 = u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$
6	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \in E_2$	$v_1 \neq v_2$
7	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 = v_2$
8	$u_1u_2 \notin E_1$	$u_1 \neq u_2$	$v_1v_2 \notin E_2$	$v_1 \neq v_2$

Therefore, the table (3) ensures that the edges sets $E(G_1 \otimes_i G_2), i \in \{0,1,...,7\}$ are mutually disjoint pairwise

Table (3): Edges Sets of New Operations

$G_1 \otimes_i G_2$	$E(G_1 \otimes_i G_2)$	$\lvert E(G_1 \otimes_i G_2) \rvert$
$G_1 \otimes_0 G_2$	$u_1u_2 \in E_1,\ v_1v_2 \in E_2$	$2q_1 q_2$
$G_1 \otimes_1 G_2$	$u_1u_2 \in E_1,\ v_1 = v_2$	$p_2 q_1$

$G_1 \otimes_i G_2$	$E(G_1 \otimes_i G_2)$	$\lvert E(G_1 \otimes_i G_2)\rvert$
$G_1 \otimes_2 G_2$	$u_1u_2 \in E_1,\ v_1v_2 \in E_2^c$	$2q_1\ q_2^c$
$G_1 \otimes_3 G_2$	$u_1 = u_2,\ v_1v_2 \in E_2$	$p_1\ q_2$
$G_1 \otimes_4 G_2$	$u_1 = u_2,\ v_1v_2 \in E_2^c$	$p_1\ q_2^c$
$G_1 \otimes_5 G_2$	$u_1u_2 \in E_1^c,\ v_1v_2 \in E_2$	$2q_1^c\ q_2$
$G_1 \otimes_6 G_2$	$u_1u_2 \in E_1^c,\ v_1 = v_2$	$p_2q_1^c$
$G_1 \otimes_7 G_2$	$u_1u_2 \in E_1^c,\ v_1v_2 \in E_2^c$	$2q_1^c\ q_2^c$

Hence, for $i, j \in \{0,1,\dots,7\}$

$$E(G_1 \otimes_i G_2) \cap E(G_1 \otimes_j G_2) = \phi \quad \text{where}$$

4.5 Remark: The new product operations generates exactly 255 different operations.

Proof: We have already seen that the classic operations can be generated by our new product operations. In fact, we can obtain many other operations by combinations of the new product operations (exactly 255), since

$$\sum_{r=1}^{8} C_r^8 = 2^8 - 1 = 255 \quad \square$$

4.6 Remark:The union of the graphs that result from all new product operations provide the complete graph

Proof: Let G_1 and G_2 be two graphs, we have to show that

$$\bigcup_{i=0}^{7}(G_1 \otimes_i G_2) = K_{p_1p_2}$$

where $|V(G_1)| = p_1$ and $|V(G_2)| = p_2$

By Definition 1.3 we have $V(\bigcup_{i=0}^{7}(G_1 \otimes_i G_2)) = \bigcup_{i=0}^{7}(G_1 \otimes_i G_2) = V_1 \times V_2$

From the table (3), we can write

$$E(\bigcup_{i=0}^{7}(G_1 \otimes_i G_2)) = (u_1,v_1)(u_2,v_2) : \begin{cases} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ u_1u_2 \in E_1, v_1 = v_2 \\ u_1u_2 \in E_1, v_1v_2 \in E_2^c \\ v_1v_2 \in E_2, u_1 = u_2 \\ v_1v_2 \in E_2^c, u_1 = u_2 \\ u_1u_2 \in E_1^c, v_1v_2 \in E_2 \\ u_1u_2 \in E_1^c, v_1 = v_2 \\ u_1u_2 \in E_1^c, v_1v_2 \in E_2^c. \end{cases}$$

It follows that

$$E(\bigcup_{i=0}^{7}(G_1 \otimes_i G_2)) = E(K_{p_1p_2})$$

Moreover by Lemma 2.1, we get $|V(\bigcup_{i=0}^{7}(G_1 \otimes_i G_2))| = |V_1| \times |V_2| = p_1 p_2$.

And by Lemma 2.2, we get

$$
\begin{aligned}
|E(\textstyle\bigcup_{i=0}^{7}(G_1 \otimes_i G_2))| &= 2q_1q_2 + p_2q_1 + p_2^2q_1 - p_2q_1 - 2q_1q_2 + p_1q_2 \\
&+ \frac{1}{2}(p_1p_2^2 - p_1p_2 - 2p_1q_2) + p_1^2q_2 - p_1q_2 - 2q_1q_2 + \frac{1}{2}(p_2p_1^2 - p_1p_2 - 2p_2q_1) \\
&+ \frac{1}{2}(p_1^2p_2^2 - p_1^2p_2 - p_1p_2^2 + p_1p_2) + (p_1q_2 + p_2q_1 - p_1^2q_2 - p_2^2q_1 + 2q_1q_2) \\
&= 2q_1q_2 + p_2q_1 + p_2^2q_1 - p_2q_1 - 2q_1q_2 + p_1q_2 + \frac{1}{2}p_1p_2^2 - \frac{1}{2}p_1p_2 - p_1q_2 \\
&+ p_1^2q_2 - p_1q_2 - 2q_1q_2 + \frac{1}{2}p_1^2p_2 - \frac{1}{2}p_1p_2 - p_2q_1 + \frac{1}{2}p_1^2p_2^2 \\
&- \frac{1}{2}p_1^2p_2 - \frac{1}{2}p_1p_2^2 + \frac{1}{2}p_1p_2 + p_2q_1 + p_1q_2 - p_1^2q_2 - p_2^2q_1 + 2q_1q_2) \\
&= \frac{1}{2}(p_1^2p_2^2 - p_1p_2) = \binom{p_1p_2}{2} = |E(K_{p_1p_2})|
\end{aligned}
$$

Therefore $$|E(\bigcup_{i=0}^{7}(G_1 \otimes_i G_2))| = |E(K_{p_1p_2})|.$$

Hence $$\bigcup_{i=0}^{7}(G_1 \otimes_i G_2) = K_{p_1p_2}. \quad \square$$

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